MA Core worksheet

# Sessions 10 and 11: Intervals and Inequalities I

Reference: Stewart Appendix A, Pages A2–A6

**Homework review:** we first spend some time going through Thursday's homework.

The main goal of these sessions is to study how to solve inequalities like

$$\frac{1}{x-2} \leq \frac{3x-1}{x-3}.$$

We will first look at intervals, then at rules for manipulating inequalities and finally at solving inequalities.

Intervals are special examples of sets of real numbers, which form line segments of the real line.

**Definition.** If $a$ and $b$ are real numbers with $a < b$ then the open interval from $a$ to $b$ is the set

$$\{x \in \mathbb{R} : a < x < b\} \qquad \text{which we denote by} \qquad (a, b).$$

The closed interval from $a$ to $b$ is the set

$$\{x \in \mathbb{R} : a \leq x \leq b\} \qquad \text{which we denote by} \qquad [a, b].$$


$(2, 4)$


$[2, 4]$

**Question.** What are simpler ways to write the intervals $(a, a)$ and $[a, a]$?

We also have half-open intervals.

**Definition.** If $a$ and $b$ are real numbers with $a < b$ then we use $[a, b)$ to denote the set

$$\{x \in \mathbb{R} : a \leq x < b\}$$

and $(a, b]$ to denote the set

$$\{x \in \mathbb{R} : a < x \leq b\}.$$


$[2, 4)$


$(2, 4]$

We also have infinite intervals.

**Definition.** If $a$ is a real number, the interval $[a, \infty)$ is the set

$$\{x \in \mathbb{R} : x \geq a\}.$$

The interval $(a, \infty)$ is the set

$$\{x \in \mathbb{R} : x > a\}.$$

The interval $(-\infty, a]$ is the set

$$\{x \in \mathbb{R} : x \leq a\}.$$

The interval $(-\infty, a)$ is the set

$$\{x \in \mathbb{R} : x < a\}.$$


Lastly the interval $(-\infty, \infty)$ is $\mathbb{R}$, the set of all real numbers.

Notice that $\infty$ is not included in any of these intervals. Remember that $\infty$ is not a number. In this situation it is just notational shorthand used to denote an interval that contains numbers as large or as small as we like.

When we use interval notation, the first number is always less than or equal to the second number. An 'interval' like $(6, 3)$ is nonsense.

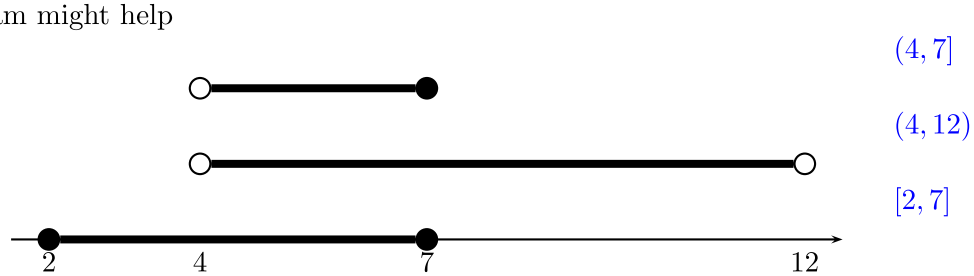
Because intervals are sets, we can compute their intersection and union.

**Example.** What is $[2, 7] \cap (4, 12]$?

We want to find all $x$ such that $2 \leq x \leq 7$ and $4 < x \leq 12$. Remember that to be in the intersection $x$, must satisfy the conditions for membership of both sets. This means that $x$ must satisfy $x > 4$ and $x \leq 7$. So the solution is

$$[2, 7] \cap (4, 12) = (4, 7].$$

A diagram might help




**Exercise.** Find

(a) $(0, 5) \cup (1, 7)$; (b) $(1, 3] \cap [2, 4)$; (c) $(-\infty, 2) \cup [1, \infty)$; (d) $(1, 2) \cap [2, 3)$.

If possible try to do these without drawing a diagram!

Order axioms for the real numbers: Just like we saw there were axioms for sets, we also have axioms for the real numbers. (Remember axioms are statements that we will take as true when we build up theory.) Two important axioms are

**Axiom 1** If $a$ and $b$ are real numbers then exactly one of the following statements is true:

$$a < b; \qquad a = b; \qquad a > b.$$

**Axiom 2** If $a$, $b$ and $c$ are real numbers such that $a < b$ and $b < c$ then $a < c$.

These two axioms are what allow us to think of $\mathbb{R}$ as a line with a real number $a$ to the left of the real number $b$ if $a < b$.

Some more familiar axioms tell us what manipulations we can make on an inequality. Compare these with what we can do with an equation.

**Axiom 3** If $a$ and $b$ are real numbers with $a < b$ and $c$ is another real number, then

$$a + c < b + c.$$

**Axiom 4** If $a$ and $b$ are real numbers with $a < b$ and $c$ is a real number with $c > 0$, then

$$ac < bc.$$

These should look similar to two of the rules for manipulating equations, except that the second axiom is only true if $c > 0$.

We can derive some more rules about manipulating inequalities from these axioms.

**Proposition 1.**

1. *If $a, b, c, d$ are real numbers with $a < b$ and $c < d$, then $a + c < b + d$.*
2. *If $a$ and $b$ are real numbers with $a < b$ and $c$ is a real number then $a - c < b - c$.*
3. *If $a$ and $b$ are real numbers with $a < b$ and $c$ is a real number with $c < 0$, then $ac > bc$.*

*Proof.* 1. Using Axiom 3 we have $a + c < b + c$ and $b + c < b + d$. Now, we can apply Axiom 2 to the numbers $a + c$, $b + c$ and $b + d$. Hence $a + c < b + d$.

2. Let $d = -c$, then using Axiom 3, $a + d < b + d$. But

$$a + d = a + (-c) = a - c \quad \text{and} \quad b + d = b + (-c) = b - c.$$

So $a - c < b - c$.

3. Let $d = -c$ again. We know that $a < b$, and $d > 0$ so using Axiom 4 applied to $a, b, d$, we have

$$da < db$$

and so

$$da - db < 0,$$

(what have we used here that justifies this?) hence

$$-db < -da.$$

But $-db = cb$ and $-da = ca$ so $cb < ca$ or equivalently $ac > bc$.

$\square$

The first part of the proposition shows that we can add inequalities together provided the direction of the inequalities is the same. The last part of this proposition is really important. If we multiply an inequality by a negative number, then the direction of the inequality changes around, i.e. $<$ must be replaced by $>$ and $>$ must be replaced by $<$.

Solving inequalities: Solving inequalities is a bit like solving equations except that the rules of what is allowed are slightly different and the solutions usually consist of one or more intervals rather than just isolated numbers.

Remember exactly what we are allowed to do to an inequality.

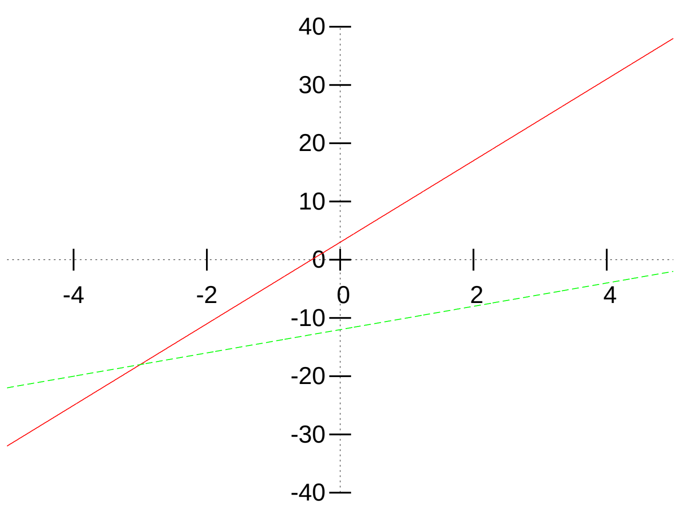
1. Add or subtract the same thing to or from both sides.
2. Multiply both sides by the same positive thing.
3. Multiply both sides by the same negative thing and reverse the direction of the inequality.

**Example.** Solve the inequality $3 + 7x < 2x - 12$.

Just like solving a simple equation, our aim is to get a single $x$ on its own on one side of the equation.

$$\begin{aligned}
3 + 7x &< 2x - 12 && \\
3 + 5x &< -12 && \text{subtracting } 2x \text{ from both sides} \\
5x &< -15 && \text{subtracting 3 from both sides} \\
x &< -3 && \text{multiplying both sides by } 1/5
\end{aligned}$$

So the solution is $(-\infty, -3)$.




**Exercise.** Solve the inequality $3x + 3 \leq 5x - 8$.

**Exercise.** Which values of $x$ satisfy both of the following inequalities at the same time

$$7 \leq 5 - 2x \qquad \text{and} \qquad 5 - 2x \leq 9?$$

**Example.** Solve the inequality $7 \leq 5 - 2x \leq 9$.

This inequality is really two inequalities

$$7 \leq 5 - 2x \qquad \text{and} \qquad 5 - 2x \leq 9.$$

We want to find the values of $x$ that satisfy both inequalities. We just saw how to solve this.

Alternatively we can work on both inequalities at once.

$$\begin{aligned} 7 &\leq 5 - 2x \leq 9 & \\ 2 &\leq -2x \leq 4 & \qquad \text{subtracting 5 from all 3 sides} \\ -1 &\geq x \geq -2 & \qquad \text{multiplying by } -1/2 \text{ and reversing both inequalities} \end{aligned}$$

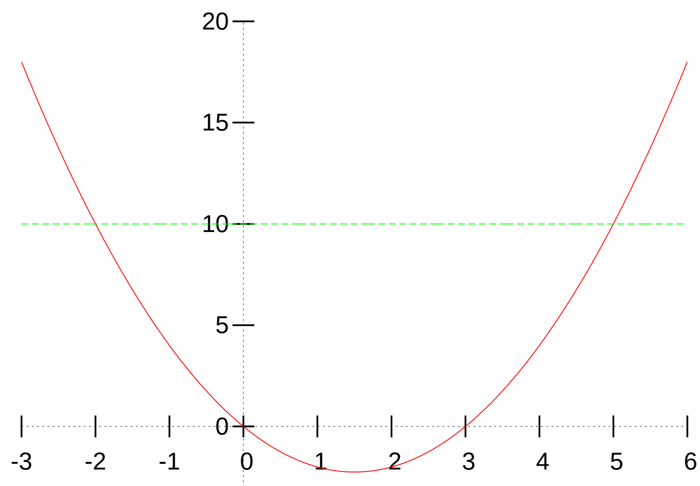
So we get the same solution $[-2, -1]$.

**Example.** Solve the inequality $x^2 - 3x \geq 10$.

This inequality is different from the previous one in two ways. First, we want the solution to include all the points where $x^2 - 3x = 10$ and second, it is a quadratic.

$$\begin{aligned} x^2 - 3x &\geq 10 & \\ x^2 - 3x - 10 &\geq 0 & \text{subracting 10 from both sides} \\ (x-5)(x+2) &\geq 0 & \text{factorising} \end{aligned}$$

We know that $(x-5)(x+2) = 0$ if $x = 5$ or if $x = -2$. We also know the shape of a quadratic curve so we know that if $x > 5$ or if $x < -2$ then $(x-5)(x+2) > 0$. So we need $x \geq 5$ or $x \leq -2$. Hence the solution is $(-\infty, -2] \cup [5, \infty)$.




**Example.** Solve the inequality $\dfrac{x-6}{x-3} \leq 0$.

To solve this inequality we just have to consider the sign of $\dfrac{x-6}{x-3}$. The sign of a fraction is determined by the sign of its numerator and denominator. For simple functions the sign can only change when the function is zero. (Actually this last fact is very subtle and needs a proof — we'll come back to look at this much more in a few weeks but we won't be able to give a proof until the analysis course next year.) So we look at when either the numerator or the denominator is zero.

$x - 6 = 0$ when $x = 6$ and $x - 3 = 0$ when $x = 3$. So the function can only change sign at $x = 3$ and $x = 6$.

Split the real line up into the intervals $(-\infty, 3)$, $(3, 6)$, $(6, \infty)$. We know that $\dfrac{x-6}{x-3}$ has the same sign on each of these intervals so we need to work out which sign it has.

To determine the sign of $\dfrac{x-6}{x-3}$ on one of these intervals, either pick a test point in the interval and see what the sign is or think about it!

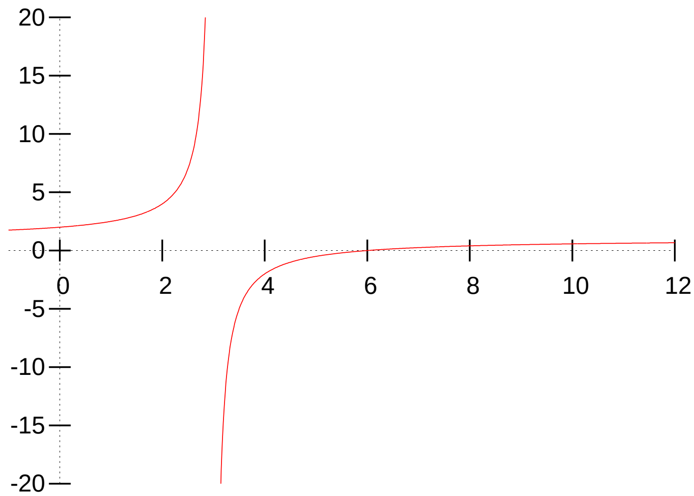
If $x \in (-\infty, 3)$ then $x - 6 < 0$ and $x - 3 < 0$ so $\dfrac{x-6}{x-3} > 0$. (Alternatively pick a test point, say 2, and check that $\dfrac{2-6}{3-6} > 0$.)

If $x \in (3, 6)$ then $x - 6 < 0$ but $x - 3 > 0$ so $\dfrac{x-6}{x-3} < 0$.

If $x \in (6, \infty)$ then $x - 6 > 0$ and $x - 3 > 0$ so $\dfrac{x-6}{x-3} > 0$.

The only points we have not considered are 3 and 6. When $x = 6$, $\dfrac{x-6}{x-3} = 0$ and when $x = 3$, $\dfrac{x-6}{x-3}$ is undefined. So $x = 3$ cannot be in the solution.

Putting all this together we see that the inequality is satisfied if $x \in (3, 6)$ or if $x = 6$. So the solution is $x \in (3, 6]$.




**Exercise.** Solve the inequalities

$$\text{(a) } x^2 + 3x - 10 < x - 2; \quad \text{(b) } \frac{2x-5}{x-2} < 1; \quad \text{(c) } -1 \leq 3x + 8 < 7.$$

## Homework for Sessions 10 and 11

1. Solve the following inequalities (writing the solution set using interval notation):

   (a) $3 \leq 4 - 2x < 7$;
   (b) $\dfrac{x}{x-4} < 3$;
   (c) $\dfrac{x}{8-x} \geq -2$;
   (d) $x^2 - 9x + 20 \leq 0$;
   (e) $x^3 - x^2 - x - 2 > 0$;
   (f) $\dfrac{1}{x+1} \geq \dfrac{3}{x-2}$;

2. Challenge question: Given that $a \leq b$, which of the following are always correct?

   (a) $a - 3 \leq b - 3$;
   (b) $-a \leq -b$;
   (c) $3 - a \leq 3 - b$;
   (d) $6a \leq 6b$;
   (e) $a^2 \leq ab$;
   (f) $a^3 \leq a^2 b$.