Sessions 12 and 13: Intervals and Inequalities II

Reference: Stewart Appendix A, Pages A6–A9

Homework review: we first spend some time going through Monday's homework.

The main goal of these sessions is to study the absolute value function and solving inequalities involving the absolute value.

Definition. The absolute value or magnitude of a real number a is denoted by $|a|$ and is the distance of a from the origin on the real line.

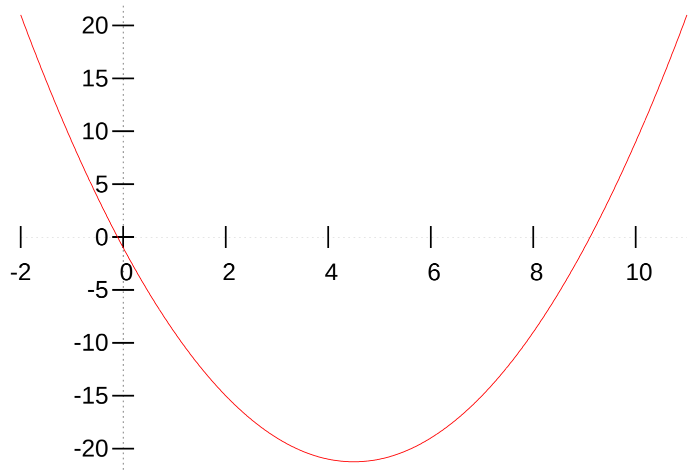
Example.

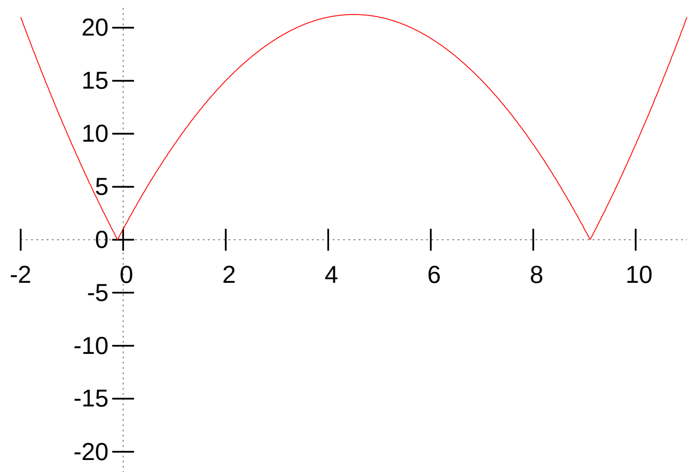
$$|2| = 2, \qquad |-2| = 2, \qquad |0| = 0, \qquad |3-\pi| = |\pi - 3|.$$

Because distances are always positive, for any real number a, we must have $|a| \geq 0$. If $a \geq 0$ then $|a| = a$ but if $a < 0$ then $|a| = -a$.

We obtain the graph of $|f(x)|$ from the graph of $f(x)$ by reflecting all the parts of the graph where $f(x) < 0$, in the x-axis.

Example. The graphs of $x^2 - 9x - 1$ and $|x^2 - 9x - 1|$ are:





Here are some properties of absolute value.

Proposition 1. *Let a and b be real numbers.*

1. $|ab| = |a||b|$;
2. $\sqrt{a^2} = |a|$;
3. $|-a| = |a|$;
4. *Providing* $b \neq 0$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

Proof. 1. We know that $|ab|$ is either equal to ab or $-ab$. We know that $|a|$ is either equal to a or $-a$. Similarly for $|b|$. So $|a||b|$ is either ab or $-ab$ but both $|ab|$ and $|a||b|$ must be non-negative so they must be equal.

2. We always have $\sqrt{a^2} \geq 0$. We split into two cases depending on whether $a \geq 0$ or $a < 0$. If $a \geq 0$ then $\sqrt{a^2} = a$ and $|a| = a$ whereas if $a < 0$ then $\sqrt{a^2} = -a$ (which is non-negative) and $|a| = -a$. So in both cases $\sqrt{a^2} = |a|$.

3. $|-a| = |(-1)a|$. Using part (1) this is equal to $|-1||a| = |a|$.

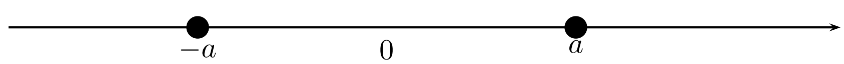
4. We omit this proof since it is almost identical to part (1).

□

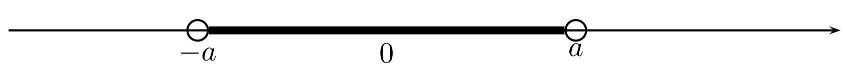
Now we see some more properties of absolute value that are used to solve inequalities.

Proposition 2. *Suppose* $a > 0$. *Then:*

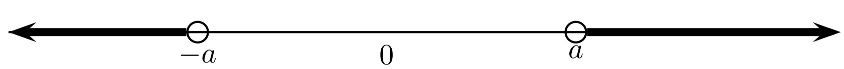
1. $|x| = a$ *if and only if* $x = a$ *or* $x = -a$;



2. $|x| < a$ *if and only if* $-a < x < a$;



3. $|x| > a$ *if and only if* $x > a$ *or* $x < -a$.



Proof. 1. If the distance along the real line from x to the origin is a then x is either a or $-a$.

2. If the distance along the real line from x to the origin is less than a then x must lie between $-a$ and a. In other words we need $x > -a$ and $x < a$, or equivalently $-a < x < a$. So here we have two inequalities and x must satisfy both.

3. If the distance along the real line from x to the origin is greater than a then x must either be greater than a or less than $-a$. Equivalently $x > a$ or $x < -a$. Here we have two inequalities and x must satisfy one or the other. (In this case it's clearly impossible for x to satisfy both inequalities.)

□

Example. Solve $|x - 5| = 8$.

Applying the first part of the proposition to $x - 5$, we see that either $x - 5 = 8$ or $x - 5 = -8$.

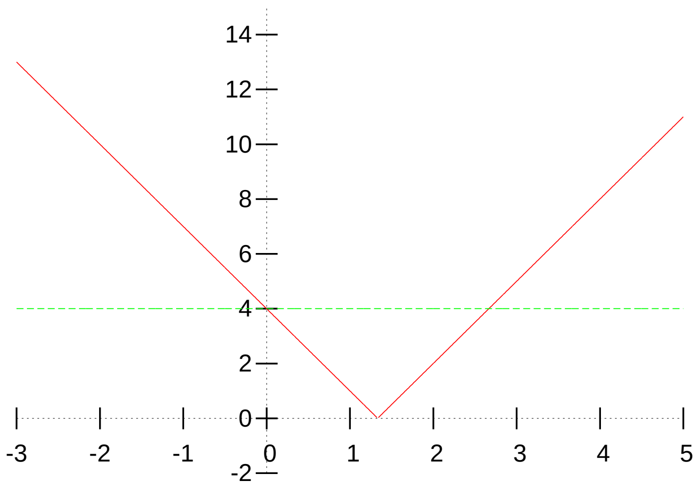
In the first case $x = 13$ and in the second case $x = -3$. So the solution is $x \in \{-3, 13\}$.

Example. Solve $|3x - 4| < 4$.

Applying the second part of the proposition to $3x-4$, we see that we must have $-4 < 3x-4 < 4$

$$\begin{aligned} -4 < 3x - 4 < 4. & \\ 0 < 3x < 8, & \qquad \text{adding 4 to all sides.} \\ 0 < x < 8/3, & \qquad \text{multiplying each side by } 1/3. \end{aligned}$$

So the solution is $x \in (0, 8/3)$.



Example. Solve $|5 - 2x| > 7$.

Applying the third part of the proposition to $5-2x$, we see that we must have either $5-2x < -7$ or $5 - 2x > 7$. We solve these two inequalities separately.

For the first inequality:

$$\begin{aligned} 5 - 2x < -7. & \\ 5 < 2x - 7, & \qquad \text{adding } 2x \text{ to both sides.} \\ 12 < 2x, & \qquad \text{adding 7 to both sides.} \\ 6 < x, & \qquad \text{multiplying both sides by } 1/2. \end{aligned}$$

So the solution to the first inequality is $x \in (6, \infty)$.

For the second inequality:

$$\begin{aligned} 5 - 2x &> 7. \\ 5 &> 2x + 7, && \text{adding } 2x \text{ to both sides.} \\ -2 &> 2x, && \text{subtracting 7 from both sides.} \\ -1 &> x, && \text{multiplying both sides by } 1/2. \end{aligned}$$

So the solution to the second inequality is $x \in (-\infty, -1)$.

We are happy if x satisfies either inequality so the solution to the original inequality is the union of the two solution sets.

Hence the solution is $x \in (-\infty, -1) \cup (6, \infty)$.

Exercise. Solve:

(a) $|4 - 2x| \leq 3$; (b) $|3x - 9| \geq 2$; (c) $\dfrac{1}{|3x + 5|} > 1$; (d) $|x^2 + 4| > 4$.

Exercise. Challenge exercise! Solve:

(a) $|x - 3|^2 - 4|x - 3| = 12$. (b) $|4x + 5| = |8x - 3|$.