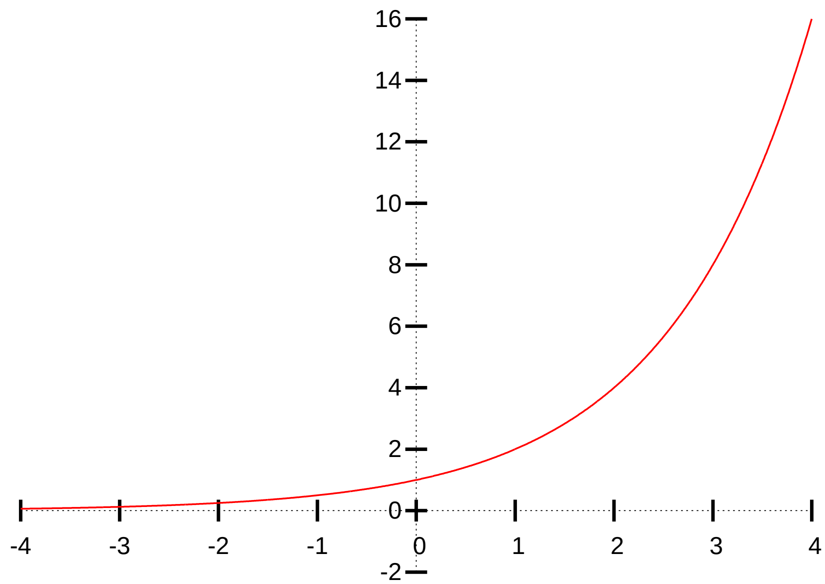
Functions XIV

Reference: Stewart Chapter 7.2, pages 392–396 and Chapter 7.3 pages 405–409.

In this session we will look at two more important functions, the exponential and logarithm functions.

Definition (Exponential Functions). An exponential function is one of the form $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = a^x$, where a is a strictly positive constant.

Example. When $a = 2$, we have $f(x) = 2^x$. Here is its graph.



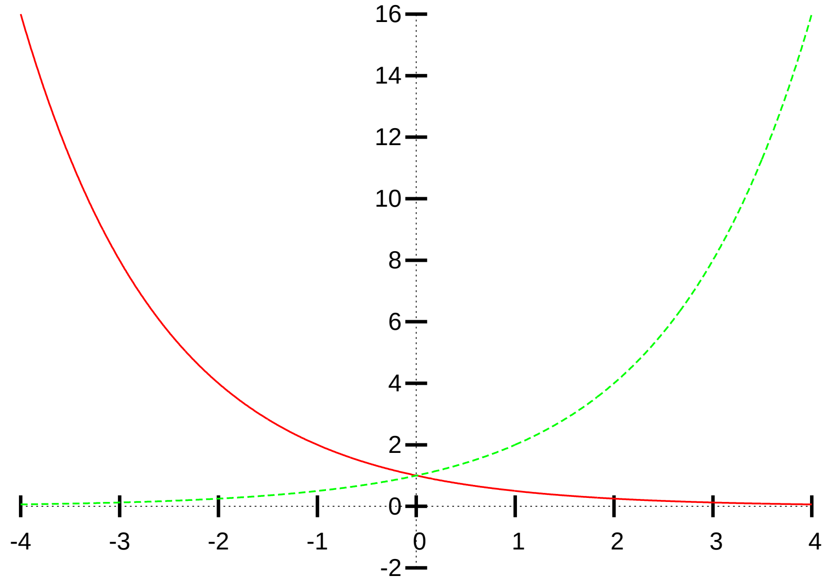
When x is a positive integer, we know how to calculate a^x.

If x is a positive rational number, say $x = p/q$ where both p and q are both positive, then $a^x = a^{p/q}$ is the qth root of a^p.

If x is a positive irrational number then it's harder to explain what a^x actually is. For the moment we will say that for irrational values of x, a^x fills in the gaps between the rational numbers so that we get a "nice smooth curve".

If x is negative then $a^x = \dfrac{1}{a^{-x}}$.

If $a = 1/2$, so the function is $f(x) = \left(\frac{1}{2}\right)^x$, then the graph is as follows.



We can see that this is formed by reflecting the previous graph in the y-axis. This should not be a surprise because if $f(x) = 2^x$ then

$$f(-x) = 2^{-x} = \left(2^{-1}\right)^x = \left(\frac{1}{2}\right)^x.$$

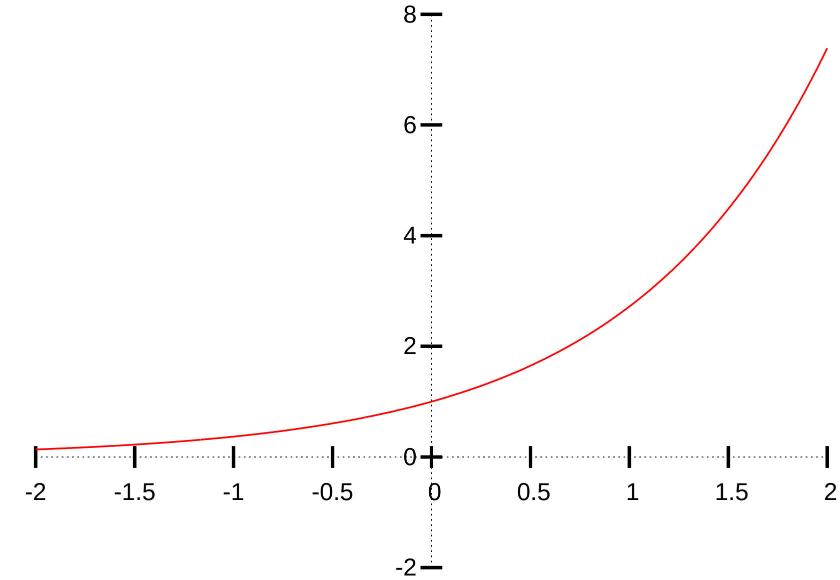
We saw earlier that the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting in the y-axis.

Proposition 1. *For all $a > 0$, $b > 0$, $x, y \in \mathbb{R}$, we have:*

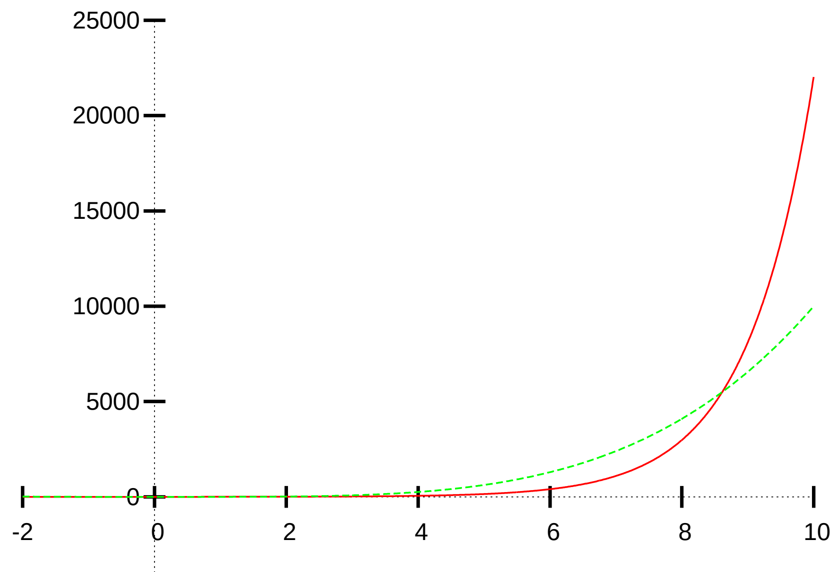
(a) $a^{x+y} = a^x a^y$; *(b)* $a^{-x} = \frac{1}{a^x}$; *(c)* $(a^x)^y = a^{xy}$; *(d)* $(ab)^x = a^x b^x$.

Proposition 2. *If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ and $a > 0$ but $a \neq 1$ then the range of f is $(0, \infty)$.*

When $a = e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828$ we get the function $f(x) = e^x$. This is called the exponential function and is one of the most important functions in mathematics. We will sometimes write it as $f(x) = \exp(x)$. Here is its graph.



Here are the graphs of $\exp$ (red) and x^4 (green), so you can see that even though $x^4 > e^x$ for some small positive values of x, $\exp(x)$ eventually becomes larger.



Definition (Logarithm)**.** If $a > 0$ and $a \neq 1$ then the function $\log_a$ is the inverse of the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = a^x$.

We need to verify that this is a reasonable definition. Why might it not be reasonable? We have to be sure that f has an inverse. But it does, because f is increasing if $a > 1$ and decreasing if $a < 1$, so this definition is actually ok.

Proposition 3. *For any $a > 0$ such that $a \neq 1$, the domain of $\log_a$ is $(0, \infty)$ and the range of $\log_a$ is $\mathbb{R}$.*

Proof. This follows from the properties of an inverse function, see Functions XIII, which tell us that if f and g are inverses of each other, then the domain of f is the range of g and the range of g is the domain of f. □

Proposition 4. *For any $a > 0$ such that $a \neq 1$ and for any k, we have*

$$\log_a 1 = 0, \qquad \log_a a = 1 \qquad \text{and} \qquad \log_a(a^k) = k.$$

Proof. Remember that $\log_a$ is the inverse of the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = a^x$.
We have $a^0 = 1$ and so $\log_a 1 = 0$. Similarly $a^1 = a$ and so $\log_a a = 1$. Finally $f(k) = a^k$ so the inverse, namely $\log_a$, must map a^k to k. □

Here are some more key properties of $\log_a$.

Proposition 5. *Let $x_1, x_2, a, b > 0$, with $a \neq 1$ and $b \neq 1$, and $c \in \mathbb{R}$. Then:*

(a) $\log_a(x_1x_2) = \log_a x_1 + \log_a x_2$; *(b)* $\log_a(x_1^c) = c\log_a(x_1)$; *(c)* $\log_b x_1 = \log_a x_1 \log_b a$.

Proof.

(a) Let $y_1 = \log_a x_1$ and $y_2 = \log_a x_2$. Then using the properties of inverse functions, $a^{y_1} = x_1$ and $a^{y_2} = x_2$. So
$$a^{y_1+y_2} = a^{y_1}a^{y_2} = x_1x_2.$$
Thus $\log_a(x_1x_2) = y_1 + y_2 = \log_a x_1 + \log_a x_2$.

(b) Similarly $a^{cy_1} = (a^{y_1})^c = x_1^c$. Thus $\log_a(x_1^c) = cy_1 = c\log_a x_1$.

(c) Let $r = \log_b x_1$ and $s = \log_b a$. Hence $x_1 = b^r$ and $a = b^s$. So
$$b^r = a^{y_1} = (b^s)^{y_1} = b^{sy_1}.$$
Hence $r = sy_1$ and consequently $\log_b x_1 = \log_b a \log_a x_1$ as required.

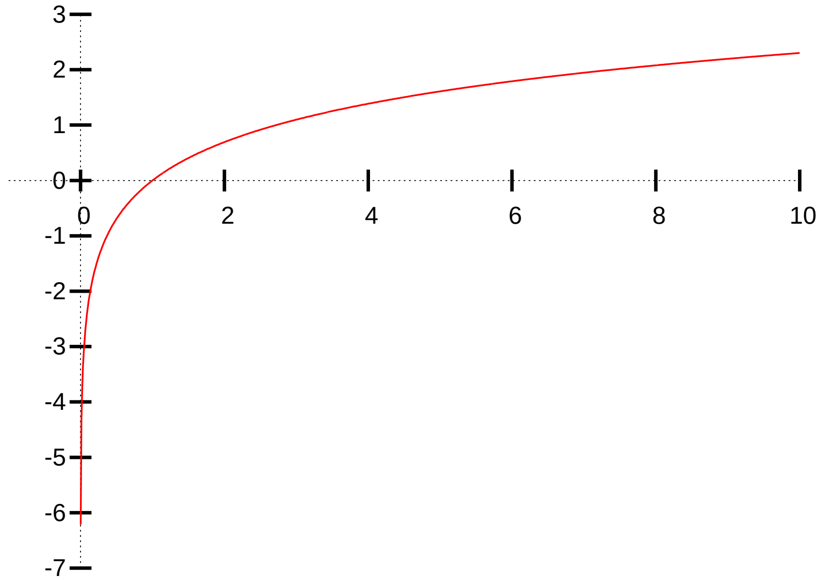
□

An important case of (b) above is when $c = -1$. This gives us $\log_a(1/x) = -\log_a x$.

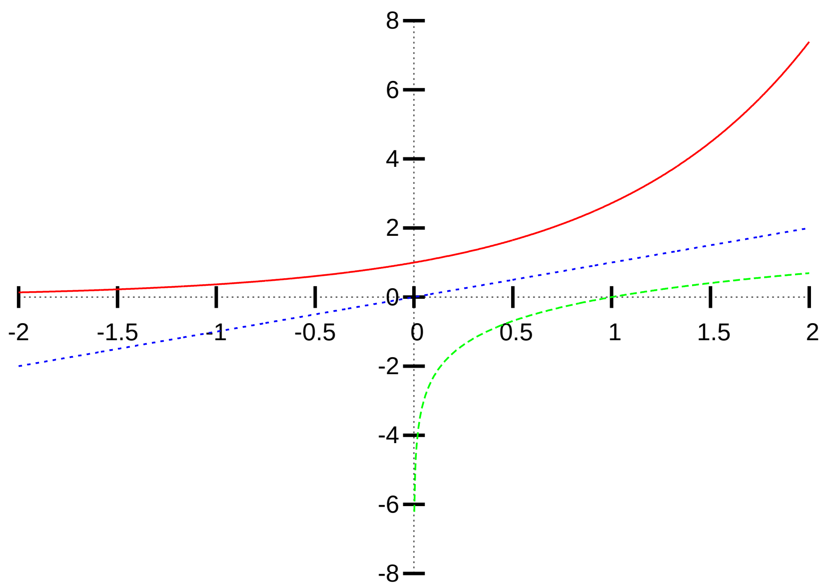
Definition (Natural Logarithm)**.** The function $\ln$ (pronounced log) or $\log$ is the inverse function of the exponential function.

So, in other words $\ln = \log_e$.

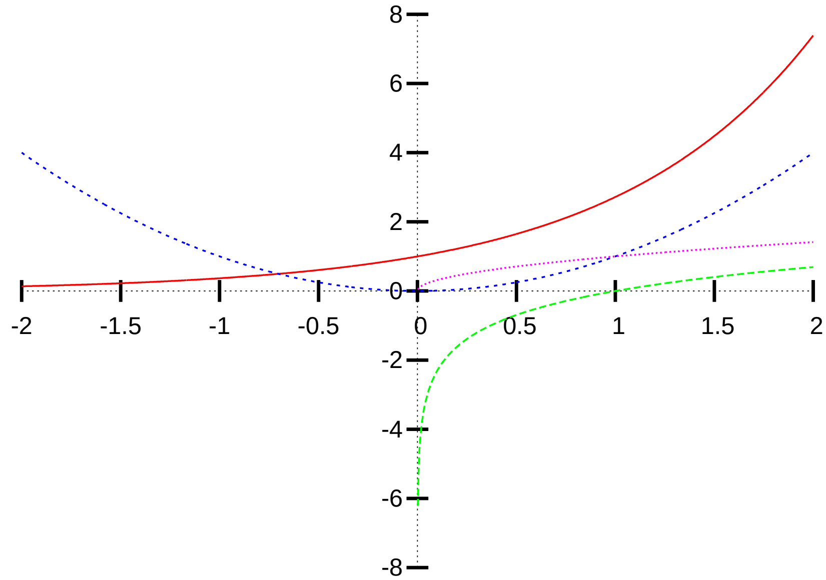
Here is the graph of ln.



Here are the two functions exp and ln on the same axes. They are symmetric about the line $y = x$ which is also shown but the different scales on the axes make it hard to see the symmetry.



On the following plot we have the functions $\sqrt{x}$ (pink), $\ln x$ (green), e^x (red) and x^2 (blue) so that it is possible to compare how the functions grow.



Example. Simplify $4 \ln 2 + \ln 3 - \ln 4$.

Solution.

$$4 \ln 2 + \ln 3 - \ln 4 = \ln(2^4) + \ln 3 - \ln 4 = \ln\left(\frac{2^4 \times 3}{4}\right) = \ln 12.$$

Example. Find x such that:

(a) $\ln(x+3) = 7$; (b) $4^x = 12$.

Solution. 1. We know that $e^{\ln(x+3)} = e^7$. So $x + 3 = e^7$ and hence $x = e^7 - 3$.

2. We have $\ln(4^x) = \ln 12$. This implies that $x \ln 4 = \ln 12$ and so $x = \dfrac{\ln 12}{\ln 4}$.

Exercise. Find all values of x such that:

(a) $\ln x^2 = 4$; (b) $\ln(1/x) = -3$; (c) $3e^{-2x} = 5$;
(d) $e^x - 3xe^x = 0$; (e) $e^x - 2e^{-x} = 4$; (f) $\ln x + \ln(2x) = 3$.

The limits of the exponential and logarithm functions are extremely important.

Theorem 1.

1. *For any a,* $\lim_{x\to a} e^x = e^a$.
2. $\lim_{x\to\infty} e^x = \infty$.
3. $\lim_{x\to -\infty} e^x = 0$.

Theorem 2.

1. *For any $a > 0$,* $\lim_{x\to a} \ln x = \ln a$.
2. $\lim_{x\to\infty} \ln x = \infty$.
3. $\lim_{x\to 0^+} \ln x = -\infty$.

Later on we will define the exponential and logarithm functions in a completely different way and we will then see how to prove this theorem.

Summary:

We have seen:

- how the exponential and logarithm functions are defined as inverses of each other;
- how to solve equations involving the exponential and logarithm functions;
- the limits of exponential and logarithm functions.