Functions XIII — Inverse Functions

Reference: Stewart Chapter 7.1.

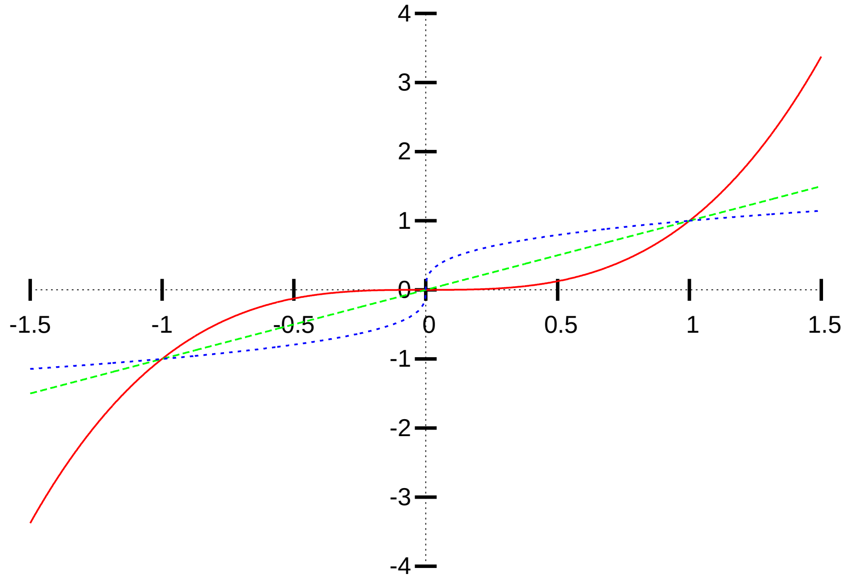
The aim of this session is to look at the definition of an inverse function and decide when they exist.

If f is a function that maps x to y, then its inverse function g maps y to x.

So $(g \circ f)(x) = g(f(x)) = g(y) = x$ and $(f \circ g)(y) = f(g(y)) = f(x) = y$.

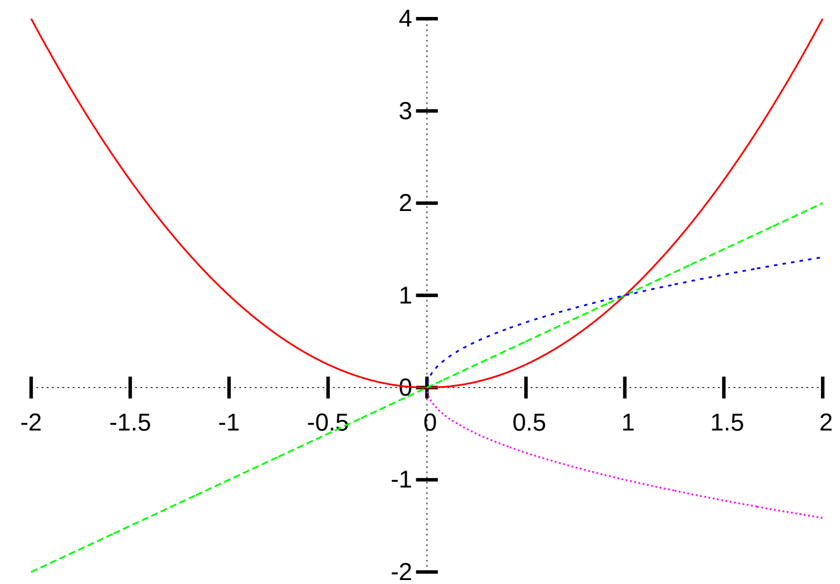
Suppose the graph of f contains the point (x, y). Then if g is the inverse of f, its graph contains the point (y, x). This means that the graph of the inverse of f is obtained from the graph of f by reflecting it in the line $x = y$.

Consider the graph of x^3 and its reflection in the line $y = x$.



The graph of the reflection is the graph of a function so this seems fine.

Now, consider the graph of x^2 and its reflection in the line $y = x$.



The graph we get by reflecting the graph of x^2 in the line $y = x$ is not the graph of a function. For instance it contains the points $(1, 1)$ and $(1, -1)$, so it cannot be the graph of a function. The problem is that $1^2 = 1$ and $(-1)^2 = 1$. We said that the inverse function of f maps y to x if f maps x to y. So where does the inverse function map 1? Is it to 1 or to -1?

This example shows that not every function has an inverse.

Let's look more formally.

Definition. If the functions f and g satisfy the two conditions

$$f(g(x)) = x \text{ for every } x \in \operatorname{dom} g$$

and

$$g(f(x)) = x \text{ for every } x \in \operatorname{dom} f$$

we say that f and g are inverse functions.

The inverse function of f is denoted by f^{-1}.

$$\boxed{f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}.}$$

Example. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^{1/3}$. Verify that f and g are inverses.

Solution. For every x, $f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$.
For every x, $g(f(x)) = g(x^3) = (x^3)^{1/3} = x$.

Exercise.

1. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = x/2$. Verify that f and g are inverses.
2. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = (x-1)^{1/3}$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^3 + 1$. Verify that f and g are inverses.

Definition. A function is one-to-one or injective if $f(x) \neq f(y)$ whenever $x \neq y$.

Example. Determine whether the following functions are one-to-one.

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$; (b) $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^3$; (c) $h : [0, \infty) \to \mathbb{R},\ h(x) = x^2$.

Solution.

(a) f is not one-to-one because $f(2) = f(-2) = 4$.

(b) g is one-to-one because it is increasing.

(c) h is one-to-one because it is increasing.

Exercise. Which of the following functions are one-to-one?

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 4 - x^2$;

(b) $g : \mathbb{R} \to \mathbb{R},\ g(x) = \cos x$;

(c) $h : (-\pi/2, \pi/2) \to \mathbb{R},\ h(x) = \cos x$;

(d) $k : (-\pi/2, \pi/2) \to \mathbb{R},\ k(x) = \sin x$.

Theorem 1. *The function f has an inverse if and only if f is one-to-one.*

Proof. Consider the function g with domain equal to the range of f and codomain equal to $\mathbb{R}$.

We define $g(x) = y$ if $f(y) = x$. This is guaranteed to be a good definition because there is a unique value of y such that $f(y) = x$. □

Proposition 1. *Suppose f has inverse f^{-1}. Then:*

1. *the domain of f^{-1} equals the range of f;*
2. *the range of f^{-1} equals the domain of f.*

Proof. We will prove the first part. The second part is left as an exercise.

The proposition states that two sets are equal, so we use the usual method of proving that the left-hand side is a subset of the right-hand side and then proving that the right-hand side is a subset of the left-hand side.

To prove that the domain of f^{-1} is a subset of the range of f:

Suppose that $x \in \mathrm{dom}(f^{-1})$. Then $f(f^{-1}(x)) = x$, by definition of the inverse function, so x is in the range of f. Hence the domain of f^{-1} is a subset of the range of f.

To prove that the range of f is a subset of the domain of f^{-1}:

Suppose that y is in the range of f. Then there is some $x \in \mathrm{dom}\, f$ such that $f(x) = y$. But $f^{-1}(f(x)) = x$ so $f^{-1}(y) = x$ which implies that $y \in \mathrm{dom}(f^{-1})$. $\square$

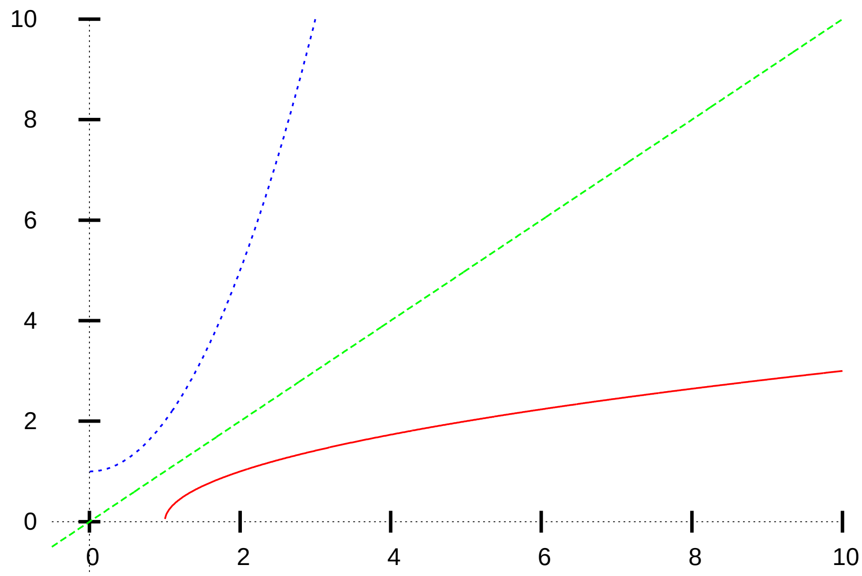
We now look at a method for finding the inverse. The idea is to solve $f(x) = y$ to obtain x as a function of y. This gives us the value of x that maps to y, which is what we want.

Example. Find the inverse of $f : [1, \infty) \to \mathbb{R},\ f(x) = \sqrt{x-1}$.

Solution. Put $y = \sqrt{x-1}$. Then $x - 1 = y^2$. So $x = 1 + y^2$. Hence $f^{-1}(y) = 1 + y^2$ and so $f^{-1}(x) = 1 + x^2$.

We must be careful to think about the domain of f^{-1} because its rule is defined for all x. However the domain is not $\mathbb{R}$.

We must use the preceding proposition which says that the domain of f^{-1} equals the range of f which is $[0, \infty)$.



Summary

We have seen:

- what the inverse of a function means;
- that not every function has an inverse;
- when a function has an inverse;
- how to find the inverse, including its domain.

Homework for Functions XIII

1. Determine whether or not the following functions are one-to-one.

   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2 - 5x$;
   (b) $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2 - 4x + 8$;
   (c) $h : [2, \infty) \to \mathbb{R},\ h(x) = x^2 - 4x + 8$;
   (d) $k : (-\pi/2, \pi/2) \to \mathbb{R},\ k(x) = \tan x$.

2. Find the inverse of the following functions, including their domains.

   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 4 - 3x$;
   (b) $g : \mathbb{R} \setminus \{2\} \to \mathbb{R},\ g(x) = \dfrac{x+2}{x-2}$;
   (c) $h : \mathbb{R} \to \mathbb{R},\ h(x) = 3 - 4x^3$.

3. Challenge question: show that the graph of $y = \dfrac{ax+b}{cx-a}$ is symmetric about the line $y = x$ providing $bc + a^2 \neq 0$.