Functions XI — Polynomials and Power Functions

Reference: Stewart Chapter 1, Pages 28–31.

The aim of this session is to look at some more important families of functions, and to think about the shapes of their graphs and some of their key properties.

Definition. A monomial is a function of the form $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax^k$ where $k \in \{0, 1, 2, \ldots\}$ and a is a constant real number.

Example. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x^3$ or $f : \mathbb{R} \to \mathbb{R},\ f(x) = \pi x^{89}$.

Definition. A polynomial is a finite sum of monomials.

So a polynomial is a function of the form $f : \mathbb{R} \to \mathbb{R},\ f(x) = a_n x^n + \ldots + a_1 x + a_0$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + \sqrt{2}x^2 - \pi x + 27$.

Definition. A rational function is a function of the form, $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials. The domain is $\{x \in \mathbb{R} : q(x) \neq 0\}$ and the codomain is $\mathbb{R}$.

Example. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \dfrac{3 - x}{x^2}$.

The main class of functions that we will consider in this session are power functions.

Definition. A power function is one of the form $f(x) = x^\alpha$ for some real number α. The codomain is $\mathbb{R}$.

The domain depends on α and is quite complicated to describe.

1. If $\alpha \in \mathbb{Q}$ and can be written as p/q where $p \in \mathbb{N} \cup \{0\}$ and q is a positive odd integer, then the domain is $\mathbb{R}$.
2. If $\alpha \in \mathbb{Q}$ and can be written as $-p/q$ where $p \in \mathbb{N}$ and q is a positive odd integer, then the domain is $\mathbb{R} \setminus \{0\}$.
3. Otherwise if $\alpha > 0$, the domain is $[0, \infty)$ and if $\alpha < 0$, the domain is $(0, \infty)$.

Exercise. Use the definition to determine the domain of the power functions corresponding to the following rules:

(a) $f(x) = x^\pi$; (b) $g(x) = x^{-3}$; (c) $h(x) = x^{-1/2}$; (d) $k(x) = x^{2/3}$.

In the remainder of this session we shall think about the graphs of these functions.

In each of the following exercises you will sketch the graphs of several functions on the same axes. You should think about the following questions:

- Which function is the greatest, second greatest, ... and when?
- What happens as x gets to be very large?
- (If appropriate) what happens as x gets to be very large in magnitude but negative?
- Which point(s) do the functions have in common?
- What happens when x is close to zero?
- Do the graphs get steeper or shallower as x gets larger or as x gets close to zero?

Exercise. Sketch the graphs of x, x^2, x^3 and x^4 on the same axes.

Come back to the following two questions later. They are only positioned here so that the pages break nicely!

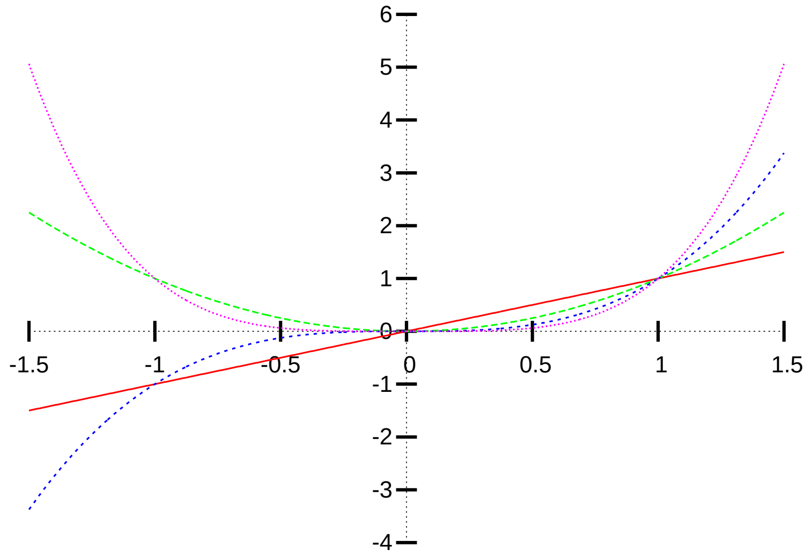
Question. Do you notice any similarity between your first and second sketches? What transformation could you apply to turn the second sketch into part of the first one.

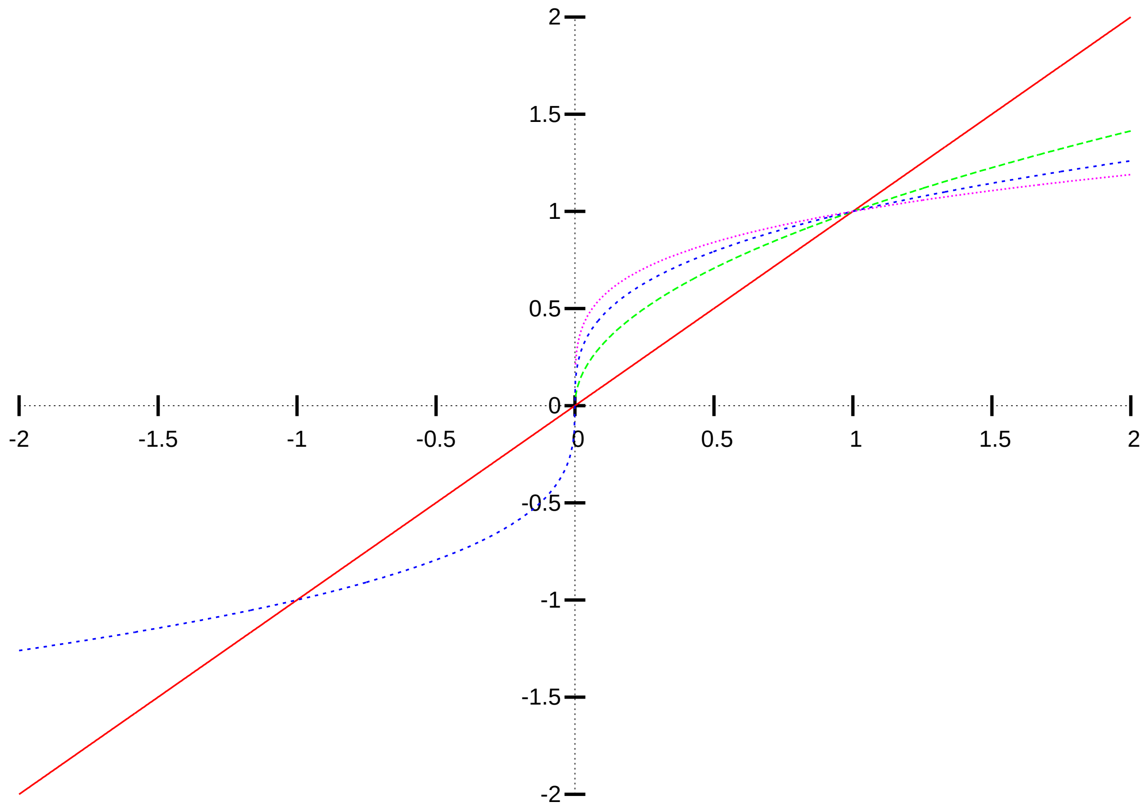
Exercise. Let $x > 0$ and $p > q$. Solve the inequality $x^p > x^q$. Does this fit with your sketches?

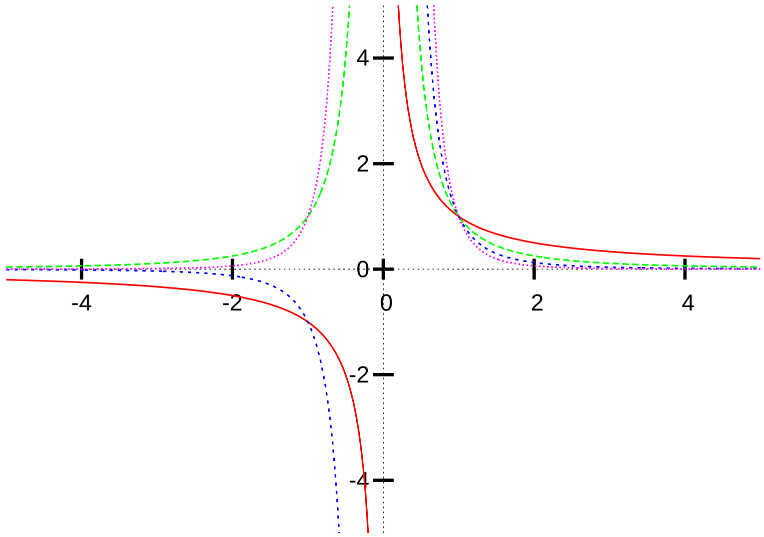
Exercise. Sketch the graphs of x, $\sqrt{x}$, $x^{1/3}$ and $x^{1/4}$ on the same axes.

Exercise. Sketch the graphs of $1/x$, $1/x^2$, $1/x^3$ and $1/x^4$ on the same axes.

The answers are as follows:







MA Core worksheet

Functions XII — Sketching Graphs

Reference: Stewart Chapter 1, Pages 37–41.

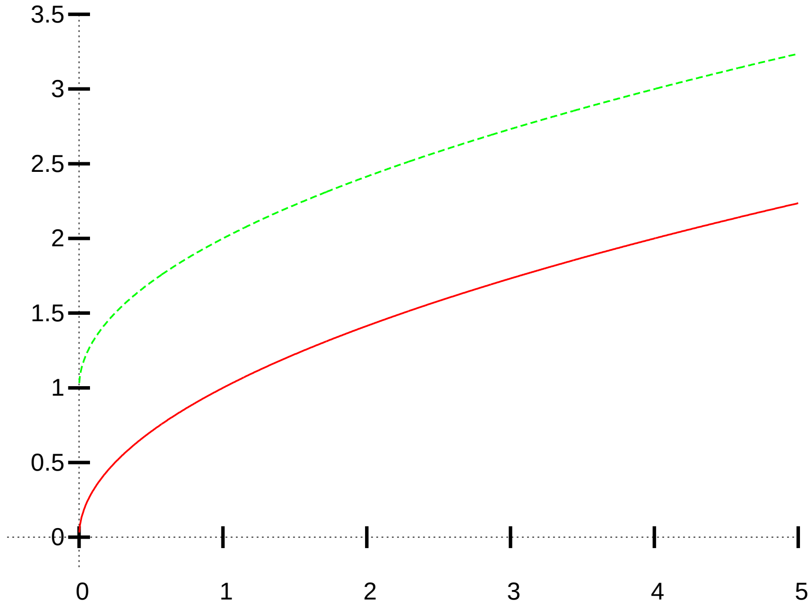
We will show how we can use the graphs of the basic functions discussed before to sketch more complicated functions by considering the effects of modifying the functions in certain ways.

The key message is that we sketch a graph by considering its basic shape and key features, perhaps by modifying the graph of a similar simpler function. We do not plot lots of points and join them up.

First recall that we have already seen two ways in which a graph may be modified.

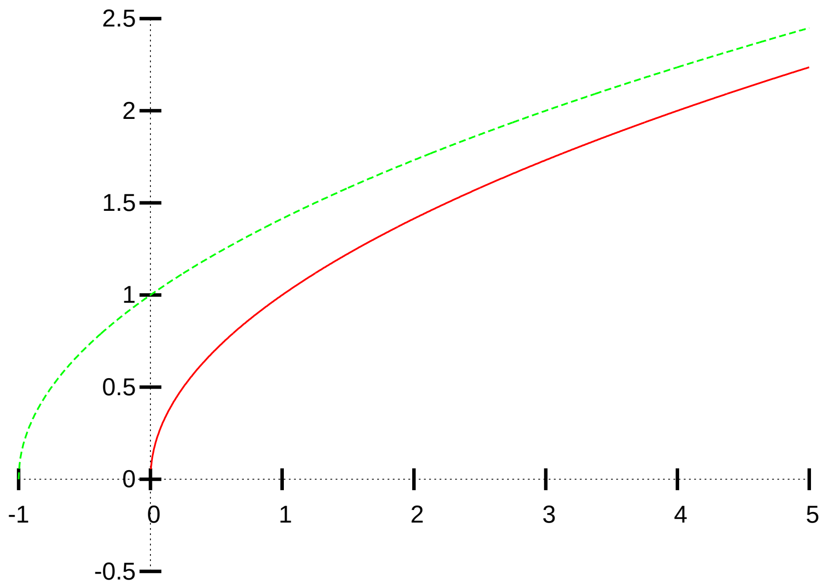
Vertical translation: if $g(x) = f(x) + A$ then the graph of g is the graph of f shifted upwards by A.

Example. The graph of $g(x) = \sqrt{x} + 1$ is obtained from the graph of $\sqrt{x}$ by shifting upwards by one unit.



Horizontal translation: if $g(x) = f(x-a)$ then the graph of g is the graph of f shifted to the right by a units.

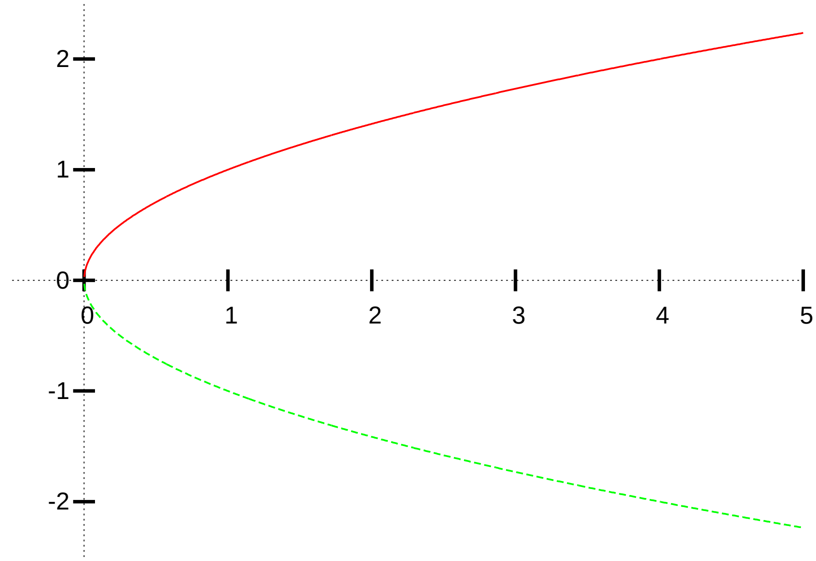
Example. The graph of $g(x) = \sqrt{x+1}$ is obtained from the graph of $\sqrt{x}$ be shifting to the left by one unit.



Two other simple modifications of a function are as follows.

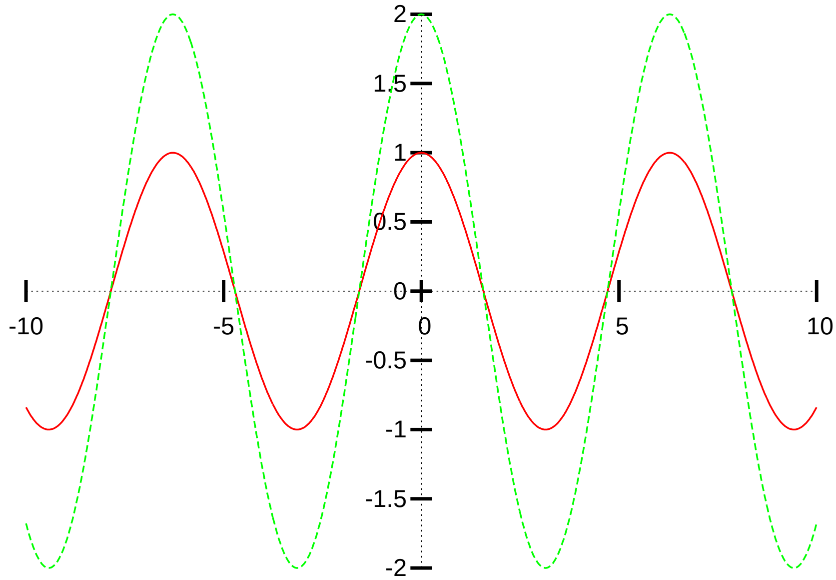
Reflection about the x-axis: if $g(x) = -f(x)$ then the graph of g is the graph of f reflected in the x-axis.

Example. The graph of $g(x) = -\sqrt{x}$ is obtained from the graph of $\sqrt{x}$ by reflecting in the x-axis.



Vertical Scaling: if $g(x) = cf(x)$ where $c > 0$, then the graph of g is the graph of f scaled by a factor of c in the vertical direction. If $c > 1$, this corresponds to stretching and if $c < 1$ this corresponds to a compression.

Example. The graph of $g(x) = 2\cos x$ is obtained from the graph of cos by a vertical scaling with a factor 2.



Exercise. Sketch on the same axes the graphs of $\sqrt{x}$ and $\sqrt{-x}$.

Exercise. Sketch on the same axes the graphs of $\sin x$, $\sin(2x)$ and $\sin(x/2)$.

Reflection about the y-axis: if $g(x) = f(-x)$ then the graph of g is the graph of f reflected in the y-axis.

Horizontal Scaling: if $g(x) = f(cx)$ where $c > 0$, then the graph of g is the graph of f scaled by a factor of $1/c$ in the horizontal direction. So the graph is stretched if $c < 1$ and compressed if $c > 1$.

Example. Sketch the graph of $y = -\sqrt[3]{8 + 8x} + 1$.

Solution. We will construct a sequence of functions that get simpler and are related by one of the transformations discussed above, until we reach a function for which we know the graph.

Start with $f(x) = -\sqrt[3]{8 + 8x} + 1$.

A simpler function related to f by a vertical translation is $g(x) = -\sqrt[3]{8 + 8x}$.

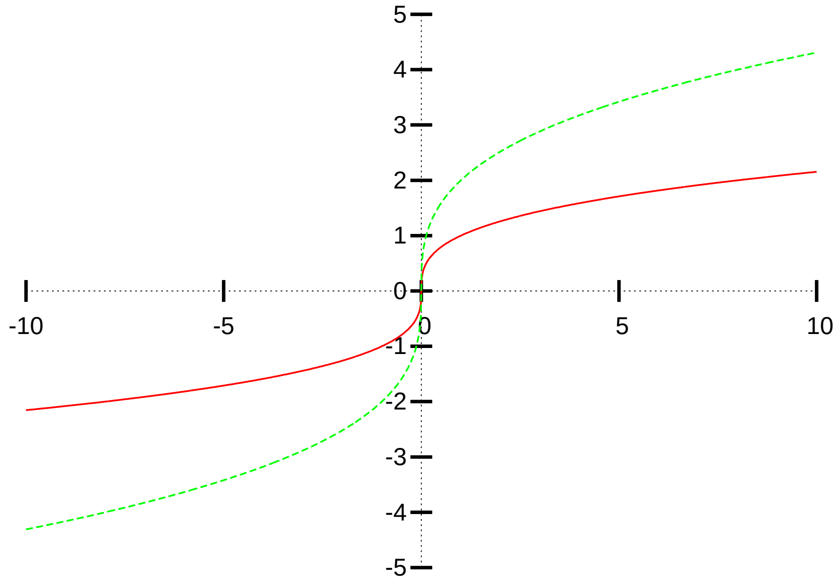
Another simpler function related to g by a reflection about the y-axis is $h(x) = \sqrt[3]{8(1 + x)}$.

Yet another simpler function related to h by a horizontal translation is $k(x) = \sqrt[3]{8x}$.

Finally we arrive at l that is related to k by a horizontal scaling and defined by $l(x) = \sqrt[3]{x}$.

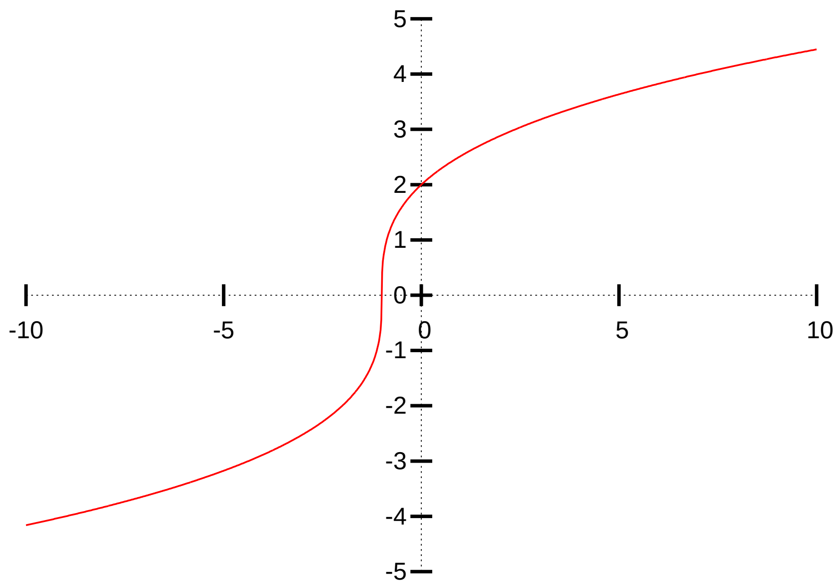
Now we construct the graphs in reverse order starting with l. The graph of k is obtained by replacing x by $8x$ so this is a horizontal scaling of factor $1/8$. So the graph is compressed.

Graphs of l (red) and k (green).



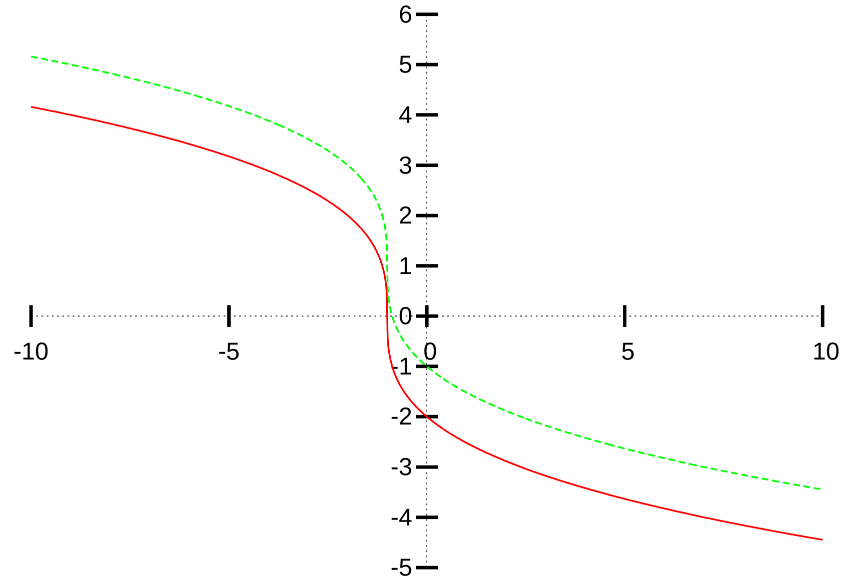
Now replace x by $1 + x$ in the definition of k, to obtain h. So h is obtained from k by a horizontal translation of one to the left.

Graph of h.



Finally to form g from h, we reflect in the x-axis and to form f we make a vertical translation of size 1.

Graph of g (red) and f (green).



Exercise. Sketch the graphs of the following functions by making translations, scalings and reflections from $|x|$, $\sqrt{x}$ and $\cos x$.

(a) $3 - |2x + 1|$; (b) $1 + \sqrt{4 - x}$; (c) $\cos(\pi - 2x)$.

Homework for Functions XI–XII

1. (a) Deduce the graph of the function $f(x) = 3 - \sqrt{x-2}$ from the graph of $\sqrt{x}$.
   (b) Deduce the graph of the function $g(x) = (2+x)^2 - 1$ from the graph of x^2.
   (c) Deduce the graph of the function $h(x) = 1 + \sin(2x - \pi)$ from the graph of $\sin x$.
   (d) Deduce the graph of the function $k(x) = \ln(x/2 - 2) + 2$ from the graph of $\ln x$.

2. Deduce the graph of the function $f(x) = \dfrac{x^2 - 4x}{x^2 - 4x + 4}$ from the graph of $1/x^2$. [Hint: Use polynomial long division.]

3. Challenge question:

   (a) Let a, b, c and d be real numbers. Show that r, s and t can be chosen so that $\dfrac{ax+b}{cx+d} = \dfrac{r}{x+s} + t$. [Hint: divide top and bottom by c.]

   (b) Describe a general method for obtaining the graph of $\dfrac{ax+b}{cx+d}$ from the graph of $1/x$.