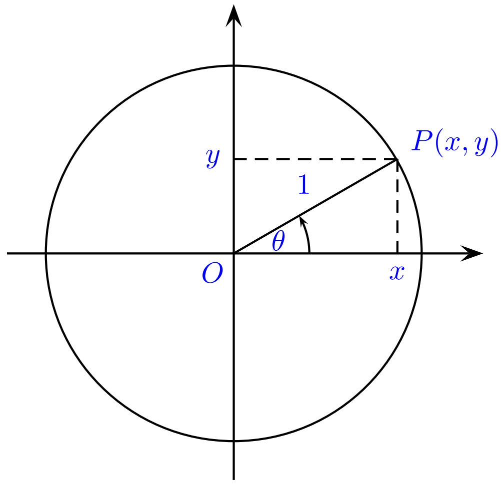
Functions IX — Trigonometric Functions II

Reference: Stewart Chapter 1, Pages 32–33 and Appendix D, Pages A26–A31.

In this session we will study more properties of the sin, cos, tan functions and introduce the sec, cosec and cot functions.

The circle in the diagram has radius 1 and its centre is at the origin. Consider the point P on the circle with co-ordinates (x, y).

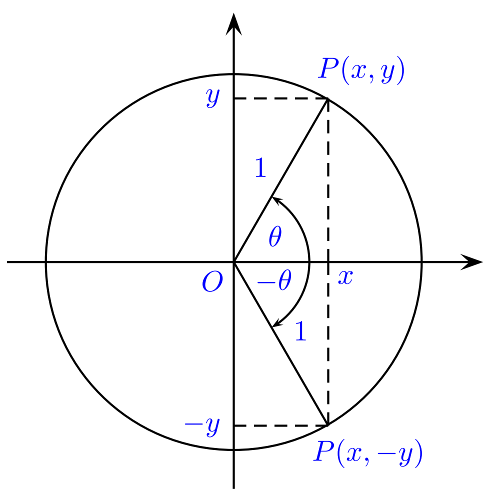


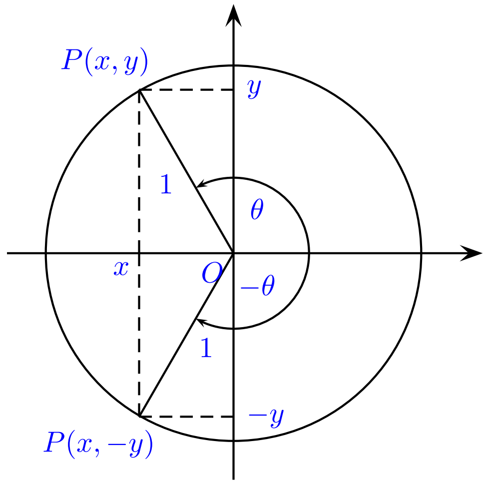
The angle θ can be positive or negative.

We can define $\cos\theta$ to be the value of x when the angle is θ.

We can define $\sin\theta$ to be the value of y when the angle is θ.

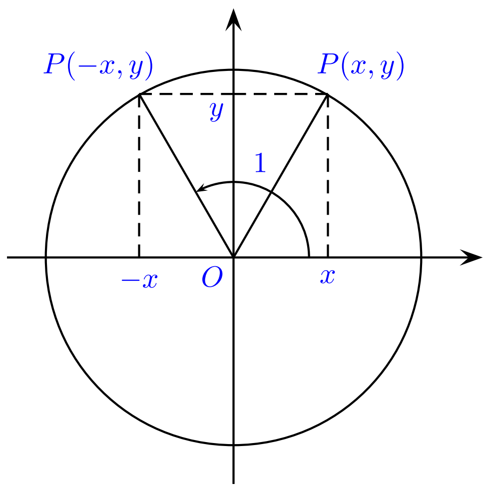
As θ increases from 0 to $\pi/2$, $\sin\theta$ increases from 0 to 1 and $\cos\theta$ decreases from 1 to 0.

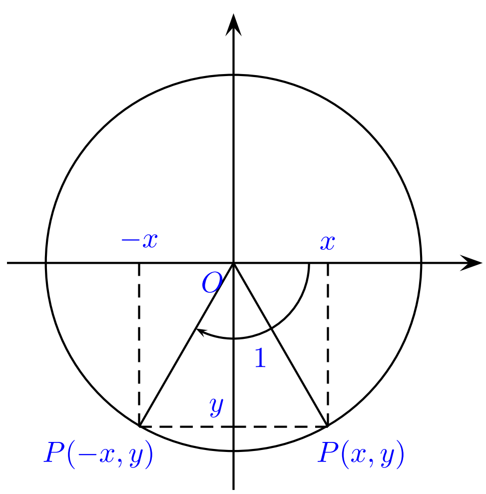




Using these two diagrams we can see that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

So cos is and sin is .





Using these two diagrams we can see that for any y with $-1 < y < 1$ there are two solutions to $\sin\theta = y$ with $\theta \in (-\pi, \pi]$.

- For both sin and cos the domain is $\mathbb{R}$ and the range is $[-1, 1]$.
- They are both periodic with period 2π, i.e. for all x

$$\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x$$

  and 2π is the smallest value for which this works.

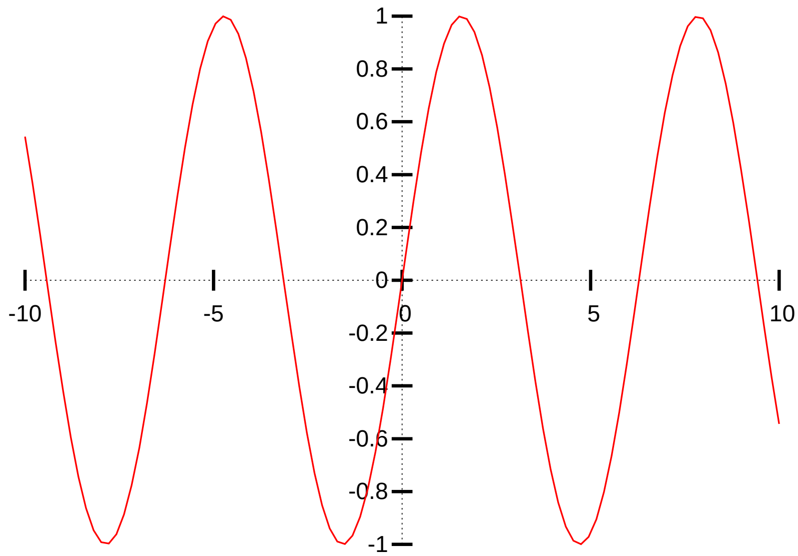
We can now introduce introduce the function tan.

- tan is defined to be $\frac{\sin}{\cos}$.
- So its domain is

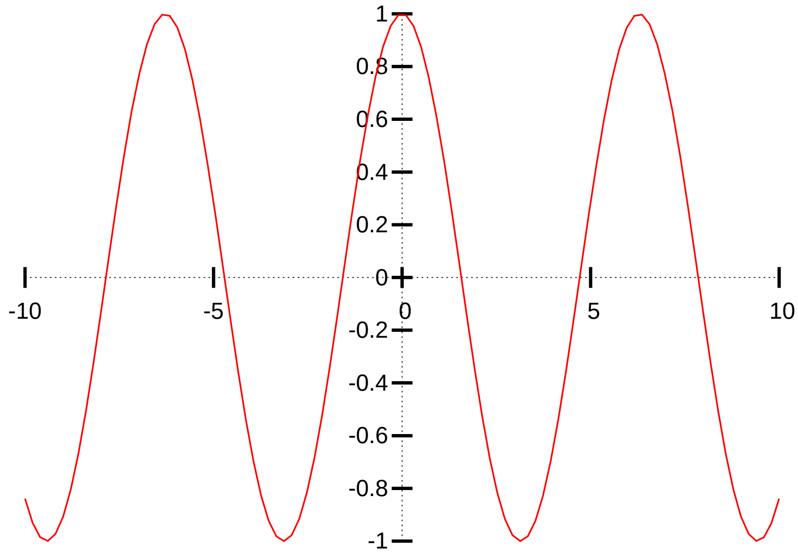
$$.$$

- $\tan(-x) = \frac{\sin(-x)}{\cos -x} = -\frac{\sin x}{\cos x} = -\tan x$, so tan is an odd function.
- As x increases from 0 to $\pi/2$, $\sin x$ increases from 0 to 1 and $\cos x$ decreases from 1 to 0, so $\tan x$ increases from 0 and tends to ∞.
- tan is periodic with period π.

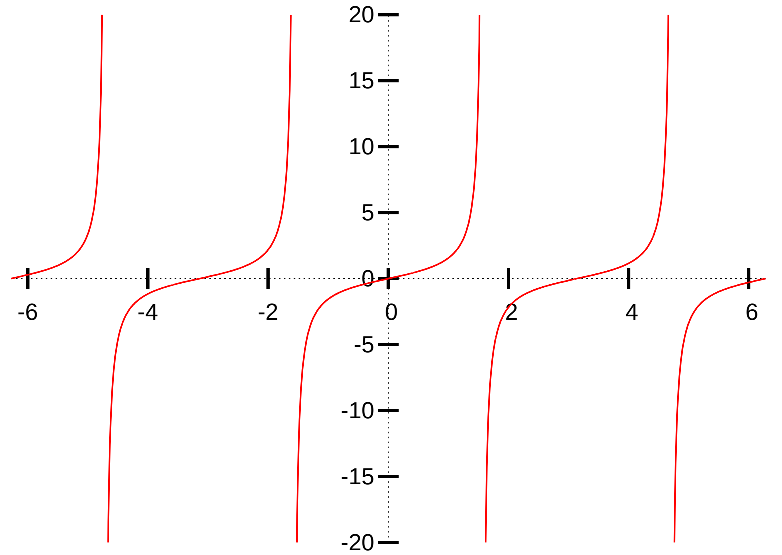
Graph of sin.



Graph of $\cos$.



Graph of $\tan$.



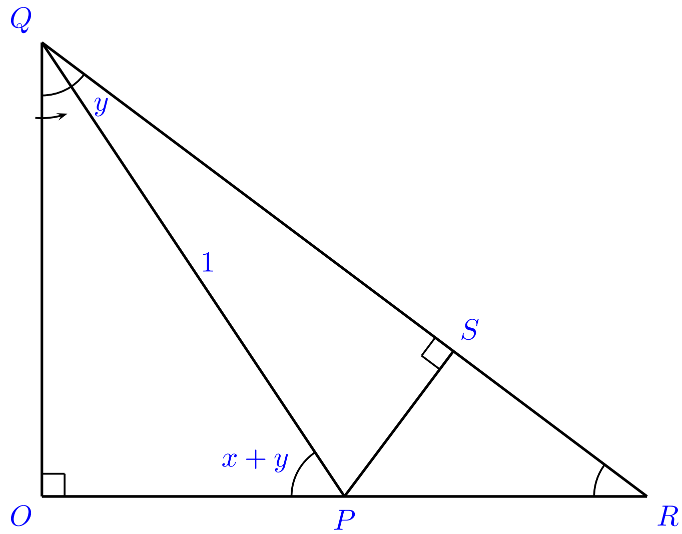
We will now introduce the familiar addition rules.

Theorem 1. *For any x and y,*

1. $\sin(x+y) = \sin x \cos y + \sin y \cos x$;
2. $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Proof. The proof of this theorem is easy using complex numbers. You will see this later in Linear Algebra.

Here is a geometrical proof that is only valid if $x > 0$, $y > 0$ and $x + y < \pi/2$.



Focus first on the triangle PQO. What is the angle PQO?

What is the length OQ?

Now look at the triangle RQP. What is the angle QRP?

Now consider the triangle PQS.

$$QS = \qquad\qquad \text{and} \quad PS =$$

Now look at the triangle SPR.

$$\sin x = \qquad\qquad\qquad \text{so} \quad PR =$$

$$\cos x = \qquad\qquad\qquad \text{so} \quad RS =$$

Finally consider the triangle ROQ.

$$\sin x = \qquad\qquad\qquad .$$

So rearranging gives

$$\sin(x+y) = \sin x\big(\sin y \cos x/\sin x + \cos y\big) = \cos x \sin y + \sin x \cos y.$$

□

We can use the addition formulae to derive the double-angle formulae.

Theorem 2. *For any x,*

1. $\sin(2x) = 2\sin x \cos x$;
2. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

Proof. Substitute $y = x$ into the addition formulae and use $\sin^2 x + \cos^2 x = 1$. □

These formulae enable us to find the sin and cos of angles using sin and cos of some known angles.

Example. Find $\cos(\pi/8)$.

Solution. Using the double angle formulae, $\cos(\pi/4) = 2\cos^2(\pi/8) - 1$. So

$$\cos^2(\pi/8) = (1 + \cos(\pi/4))/2 = (2 + \sqrt{2})/4.$$

Hence $\cos(\pi/8) = \dfrac{\sqrt{2+\sqrt{2}}}{2}$.

Exercise. Find:

(a) $\sin(\pi/8)$; (b) $\cos(\pi/12)$; (c) $\cos(\pi/16)$.

We also easily derive the subtraction rules.

Theorem 3. *For any x and y,*

1. $\sin(x - y) = \sin x \cos y - \cos x \sin y$;
2. $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

Proof. Using the addition rule,

$$\begin{aligned}\sin(x - y) = \sin(x + (-y)) &= \sin x \cos(-y) + \cos x \sin(-y)\\ &= \sin x \cos y - \cos x \sin y.\end{aligned}$$

You do the other part! □

The rules we have seen so far enable us to derive some well-known trigonometric identities.

Exercise. Find expressions for:

(a) $\cos\left(\frac{\pi}{2} - \theta\right)$; (b) $\sin\left(\frac{\pi}{2} - \theta\right)$; (c) $\sin(\pi - \theta)$;

(d) $\sin(\pi + \theta)$; (e) $\cos(\pi - \theta)$; (f) $\cos(\pi + \theta)$;

Remember that we said if $-1 < x < 1$ then:

- there are two values of $\theta \in (-\pi, \pi]$ such that $\sin\theta = x$;
- there are two values of $\theta \in (-\pi, \pi]$ such that $\cos\theta = x$;

More precisely we have,

Theorem 4.

- *If θ is one solution of $\sin\theta = x$ then the set of all solutions is*

$$\{\theta + 2n\pi, \pi - \theta + 2n\pi : n \in \mathbb{Z}\}.$$

- *If θ is one solution of $\cos\theta = x$ then the set of all solutions is*

$$\{\theta + 2n\pi, -\theta + 2n\pi : n \in \mathbb{Z}\}.$$

- *If θ is one solution of $\tan\theta = x$ then the set of all solutions is*

$$\{\theta + n\pi : n \in \mathbb{Z}\}.$$

Summary so far:

We have seen:

- How we can define the trigonometric functions using the point moving on the circle and derive their properties.
- How we can see properties of the trigonometric functions from their graphs.
- Addition and subtraction rules and how to use them.
- How to solve trigonometric equations.

MA Core worksheet

Functions X — Trigonometric Functions III

We can now introduce the functions sec, cosec and cot.

Definition.

- sec is defined to be $1/\cos$;
- cosec is defined to be $1/\sin$;
- cot is defined to be $\cos/\sin$.

So the domain of sec is

.

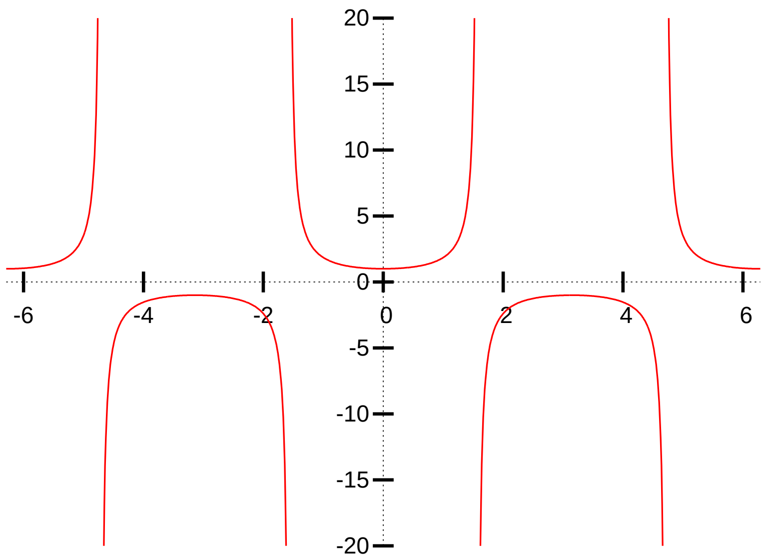
The domain of cosec and cot is

.

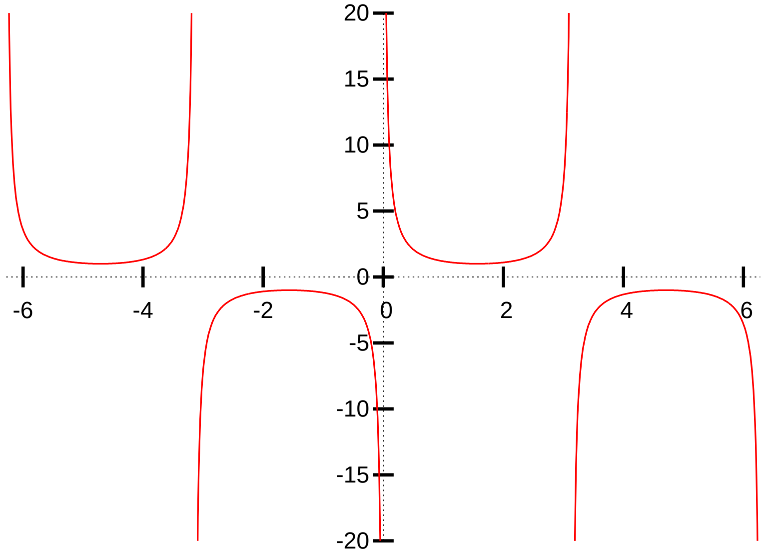
Because the range of sin and cos is $[-1, 1]$, the range of sec and cosec is $\mathbb{R} \setminus (-1, 1)$.

The range of cot is $\mathbb{R}$.

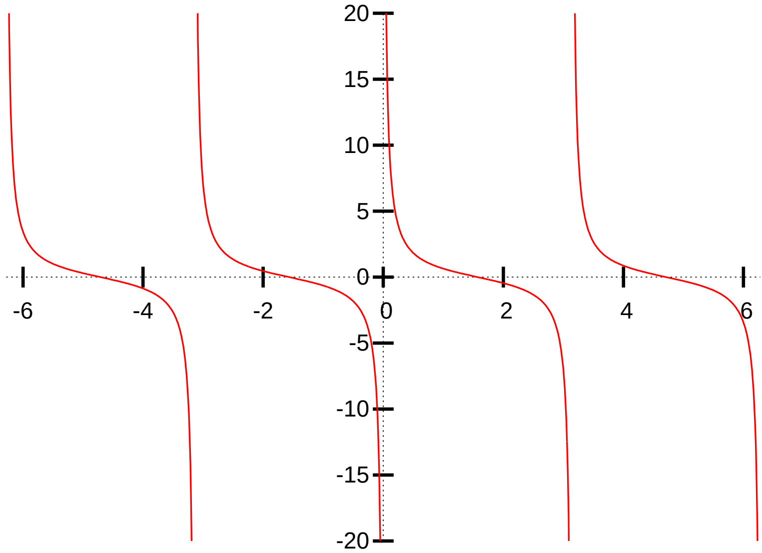
Graph of sec.



Graph of cosec.



Graph of cot.



We can now prove the following theorem.

Theorem 5.

1. *For all* x,

$$\cos^2 x + \sin^2 x = 1.$$

2. *For all* $x \in \mathbb{R} \setminus \{\pi/2 + n\pi : n \in \mathbb{Z}\}$,

$$1 + \tan^2 x = \sec^2 x.$$

3. For all $x \in \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$,

$$1 + \cot^2 x = \operatorname{cosec}^2 x.$$

Proof.

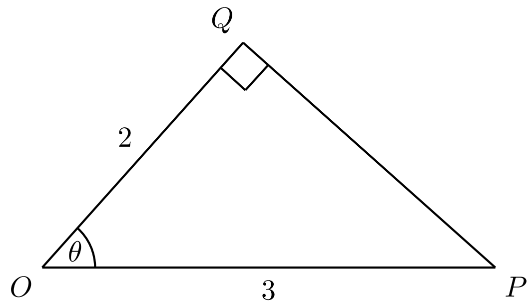
1. We proved this in trigonometry I. It is essentially Pythagoras's Theorem.
2. Divide the equation in part (a) by $\cos^2 x$.
3. Divide the equation in part (b) by $\sin^2 x$.

□

Sometimes we know one of the six trigonometric functions and we want to know the other five.

Example. Suppose we know that $\cos\theta = 2/3$ and that $0 \le \theta \le \pi/2$. Find $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$.

Solution. Draw a right-angled triangle including an angle θ such that $\cos\theta = 2/3$.



Using Pythagoras's theorem, the side PQ has length . Hence

.

Exercise. 1. Suppose we know that $\sin\theta = 3/5$ and that $0 \le \theta \le \pi/2$. Find the values of the other five trigonometric functions.

2. Suppose we know that $\tan\theta = 1/2$ and that $0 \le \theta \le \pi/2$. Find the values of the other five trigonometric functions.

Summary so far:

We have seen:

- Relationships like $\sin^2\theta + \cos^2\theta = 1$.
- Given the value of one trigonometric function, how to find the values of the others.

MA Core worksheet

Homework for Functions IX–X

1. Finish off any in-class exercises remaining.

2. Determine whether the following functions are even or odd or neither:

   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^4 + x \sin x$; (b) $g : (-\pi/2, \pi/2) \to \mathbb{R},\ g(x) = \tan x + x \cos x$.

3. Prove that:

   (a) $\sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$;

   (b) $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$;

   (c) $\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$.

4. Find all values of θ such that:

   (a) $\sec \theta = 2/\sqrt{3}$; (b) $\operatorname{cosec} \theta = -2$; (c) $\cot \theta = \sqrt{3}$.

5. Prove that:

   (a) $\dfrac{\cos\theta \sec\theta}{1 + \tan^2\theta} = \cos^2\theta$; (b) $2\operatorname{cosec}(2\theta) = \sec\theta \operatorname{cosec}\theta$;

   (c) $\tan\theta + \cot\theta = 2\operatorname{cosec}(2\theta)$; (d) $\cos\left(\frac{\pi}{3} + \theta\right) + \cos\left(\frac{\pi}{3} - \theta\right) = \cos\theta$.

6. Use the addition rules for sin and cos to prove that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

7. (a) Suppose that $\sin\theta = 1/3$ and that $0 \le \theta \le \pi/2$. Find the values of the other five trigonometric functions.

   (b) Suppose that $\sin\theta = 3/4$ and that $\pi/2 \le \theta \le \pi$. Find the values of the other five trigonometric functions.

   (c) Suppose that $\sin\theta = -2/3$ and $\cos\theta = \sqrt{5}/3$. Find the values of the other four trigonometric functions.