Functions VI — Compositions

Reference: Stewart Chapter 1, Pages 41–43.

Earlier on, we saw that we could combine functions by adding, subtracting, multiplying or dividing them. In this session we will study combining functions in another way, by composing them.

If we have two functions f and g then we can define a new function by first applying f to the input x and then applying g to the answer $f(x)$ that we get. This gives us $g(f(x))$.

Example. If $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$ and $f : \mathbb{R} \to \mathbb{R},\ f(x) = x + 1$, what is $g(f(3))$?

Solution. $f(3) = 3 + 1 = 4$ and then $g(f(3)) = g(4) = 16$.

Exercise. Find $g(f(a))$ when f, g, a are as given below:

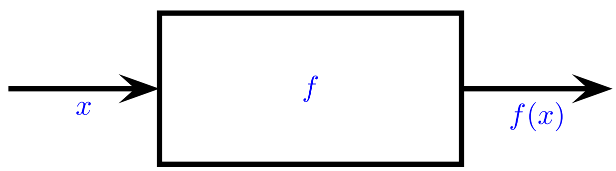
(a) $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2 + 1,\ f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = 1/x,\ a = 2;$

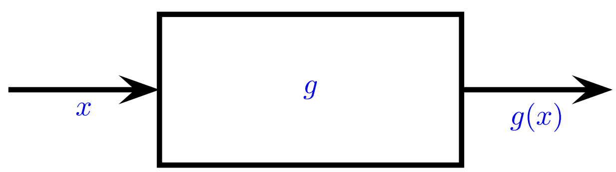
(b) $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2,\ f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x,\ a = \pi/6;$

(c) $g : \mathbb{R} \to \mathbb{R},\ g(x) = |x|,\ f : \mathbb{R} \to \mathbb{R},\ f(x) = x - 4,\ a = 2;$

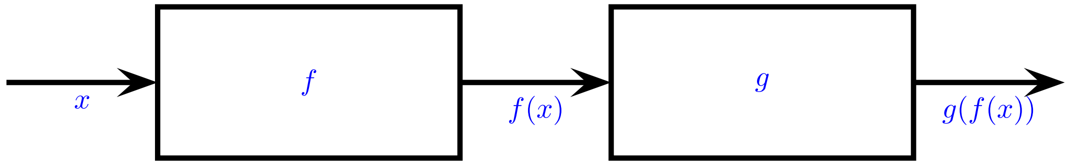
(d) $g : [0, \infty) \to \mathbb{R},\ g(x) = \sqrt{x},\ f : \mathbb{R} \to \mathbb{R},\ f(x) = x - 5,\ a = 2.$

If we depict the two functions f and g separately as machines.





Then when we compose them we get the following picture.



The key point is that we are applying f to the output $g(x)$ of the first function g.

Formally given two functions f and g their composition $g \circ f$ is the function with rule

$$(g \circ f)(x) = g(f(x)),$$

and domain

$$\{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}.$$

We obtain the rule by substituting $f(x)$ in place of x in the definition of $g(x)$.

x is in the domain of $g \circ f$ if and only if it is in the domain of f and $f(x)$ is in the domain of g. Does this make sense given the picture above? Does this explain what was happening in the last part of the first exercise?

Example. Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ and $g : \mathbb{R} \setminus \{1\} \to \mathbb{R},\ g(x) = 1/(x-1)$. Find $g \circ f$.

Solution. The rule is $(g \circ f)(x) = g(f(x)) = 1/(x^2 - 1)$.

The domain is

$$\{x \in \mathrm{dom}\, f : f(x) \in \mathrm{dom}\, g\} = \{x \in \mathbb{R} : x^2 \in \mathbb{R} \setminus \{1\}\} = \{x \in \mathbb{R} : x^2 \neq 1\} = \mathbb{R} \setminus \{-1, 1\}.$$

Exercise. Given f and g, find $g \circ f$.

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 3$ and $g : [3, \infty) \to \mathbb{R},\ g(x) = \sqrt{x-3}$;

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ and $g : [1, \infty) \to \mathbb{R},\ g(x) = \sqrt{x-1}$;

(c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{x-1}{1+x^2}$ and $g : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ g(x) = \dfrac{1}{x}$.

It is also important to be able to spot that a function is a composition and be able to split it up into its constituent parts.

Exercise. For each of the functions f below, express f as a composition. In other words, find g and h so that $f = g \circ h$.

(a) $f(x) = \sqrt{x+2}$; (b) $f(x) = |x^2 - 2x - 8|$; (c) $f(x) = \dfrac{3}{5 + \cos x}$; (d) $f(x) = 3 \sin x^2$.

Functions VII — Odd and Even Functions

Reference: Stewart Chapter 1, Page 19.

In this session we will study two key properties, which functions may have, concerned with symmetry . These will turn out to be really useful for understanding properties of some important functions and in simplifying calculations.

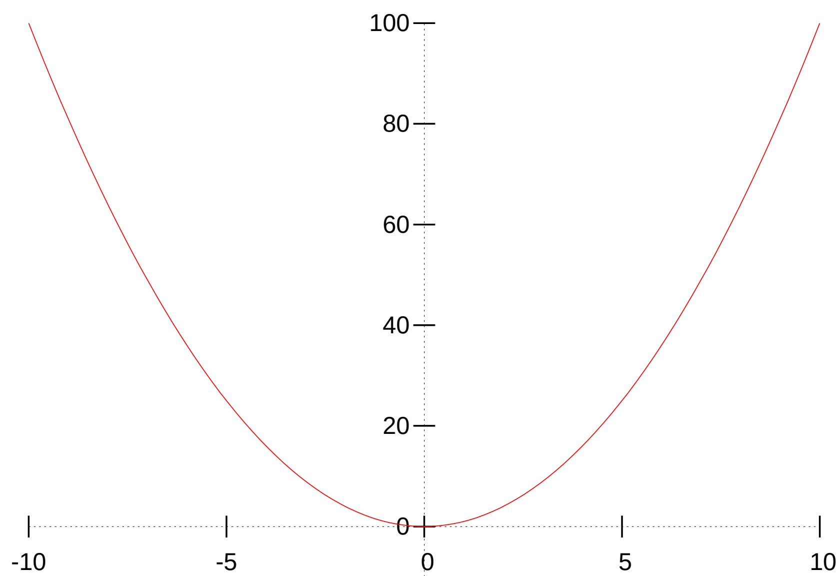
Definition. A function f is *even* if for all $x \in \operatorname{dom} f$, $f(-x) = f(x)$; we say that a function f is *odd* if for all $x \in \operatorname{dom} f$, $f(-x) = -f(x)$.

A function being even or odd has nothing to do with being divisible by two.

The graph of an even function is symmetric about the y-axis. The graph of an odd function is symmetric about the origin.

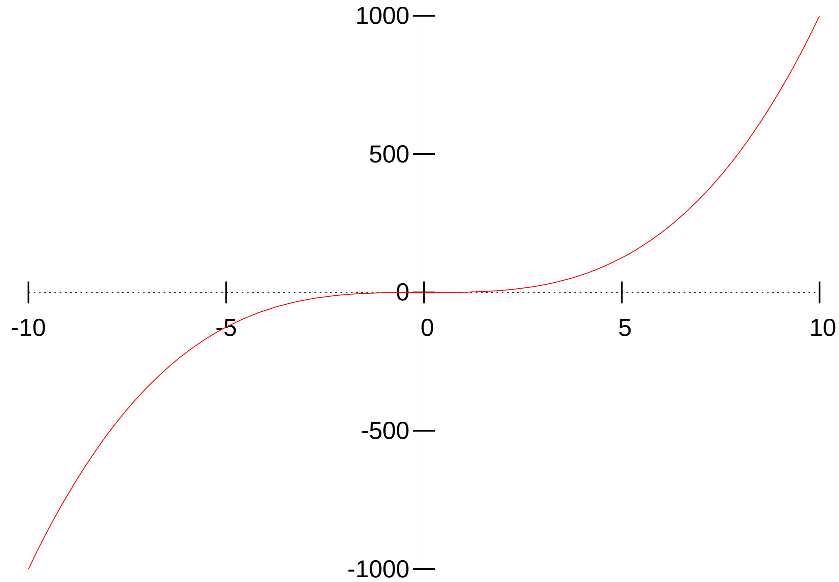
Example. 1. The function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$



2. The function $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^3$ is odd because

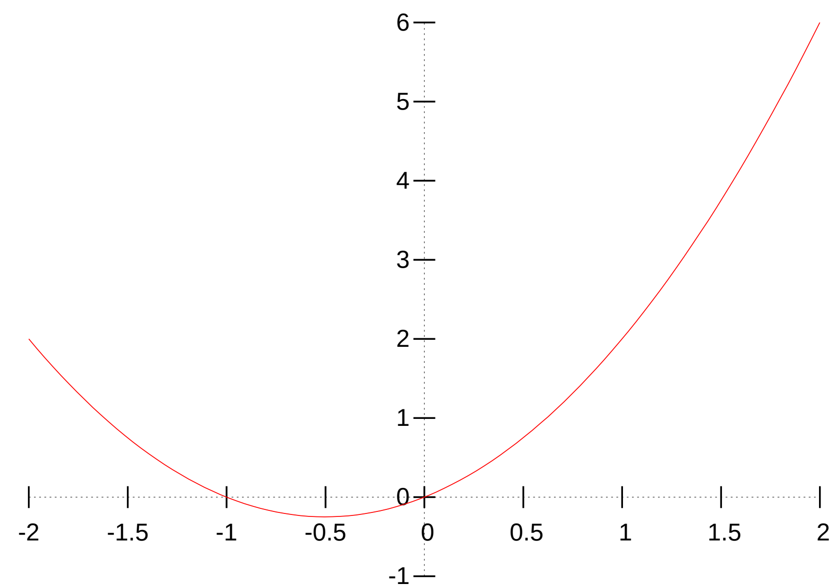
$$g(-x) = (-x)^3 = -x^3 = -g(x).$$



3. The function $h : \mathbb{R} \to \mathbb{R},\ h(x) = x^2 + x$ is neither even nor odd because

$$h(1) = 2 \text{ and } h(-1) = 0$$

so $h(1) \neq h(-1)$ and $h(1) \neq -h(-1)$.



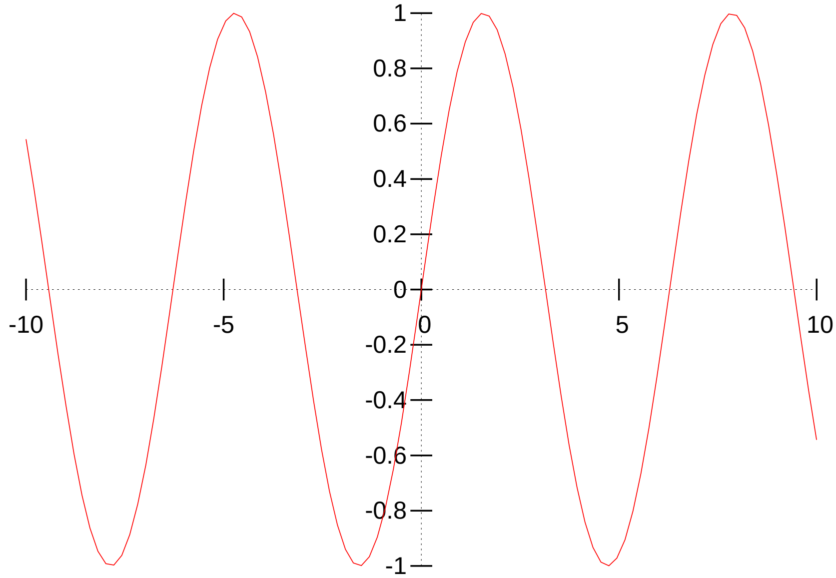
To prove that a function is even we must find an expression for $f(-x)$ and show that it is equal to $f(x)$.

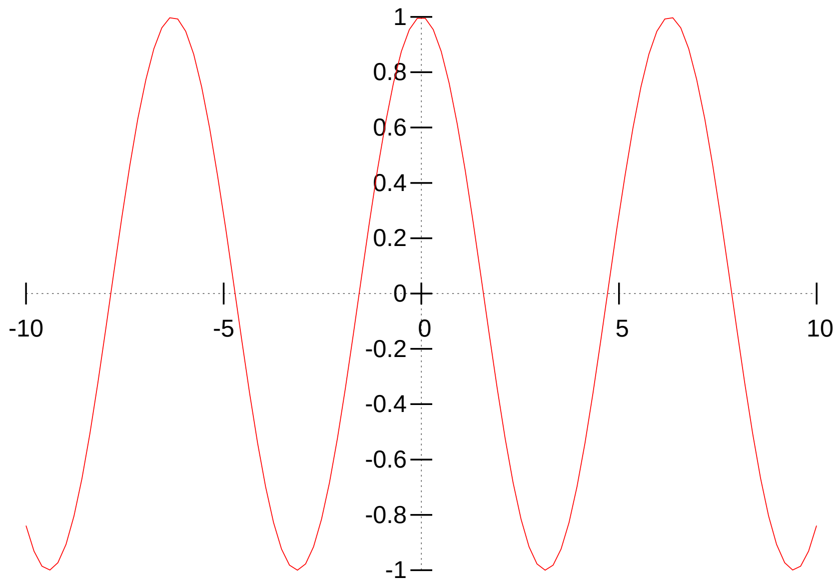
To prove that a function is odd we must find an expression for $f(-x)$ and show that it is equal to $-f(x)$.

To prove that a function is not even we just need to find one number a with $f(-a) \neq f(a)$. If there is one value a where $f(-a) \neq f(a)$ then it can't be true that for every value x, $f(-x) = f(x)$.

To prove that a function is not odd we just need to find one number a with $f(-a) \neq -f(a)$. If there is one value a where $f(-a) \neq -f(a)$ then it can't be true that for every value x, $f(-x) = -f(x)$.

Example. We can see from the graphs of sin and cos that sin is odd and cos is even. We can prove this properly later.





Exercise. Determine which of the following are odd and which are even.

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = |x|$;

(b) $g : \mathbb{R} \to \mathbb{R},\ g(x) = 1 + x$;

(c) $h : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ h(x) = 1/x$;

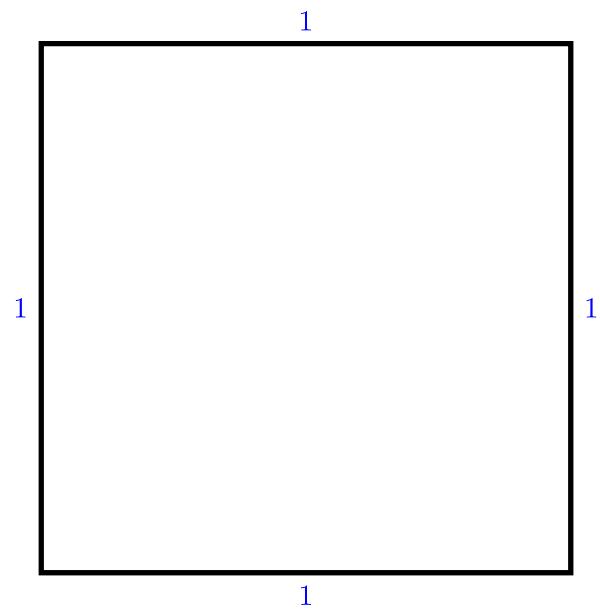
(d) $k : \mathbb{R} \to \mathbb{R},\ k(x) = x^3 + x^2$.

Functions VIII — Trigonometric Functions I

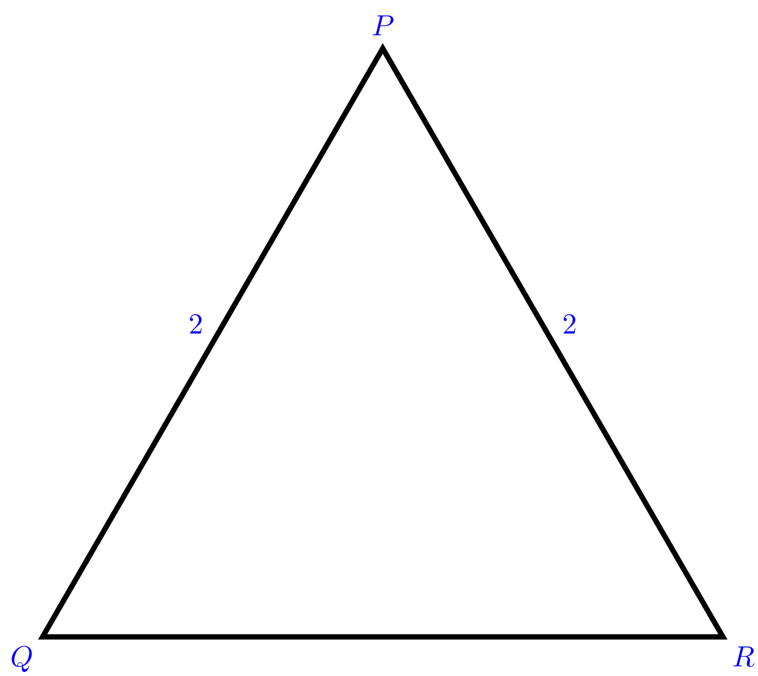
Reference: Stewart Chapter 1, Pages 32–33 and Appendix D, Pages A26–A28.

In this section we will study basic properties of the sin, cos and tan functions.

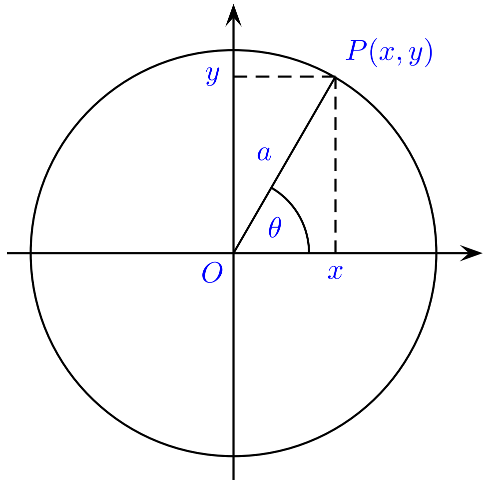
Exercise. In the square below, draw in one of the diagonals, mark in the size of all the angles and all the remaining lengths. Use the diagram to work out $\sin \pi/4$ and $\cos \pi/4$.



Exercise. In the equilateral triangle below, draw in the angle bisector from vertex P to the line QR. Mark in the size of all the angles and all the remaining lengths. Use the diagram to work out $\sin \pi/6$, $\sin \pi/3$, $\cos \pi/6$ and $\cos \pi/3$.



Exercise. The circle in the diagram has radius a and its centre is at the origin. Consider the point P on the circle with co-ordinates (x, y).



1. What is the relationship between x, and a and θ?

2. What is the relationship between y, and a and θ?

3. What is the relationship between x, y and a?

4. What happens to x and y as θ increases from 0 to $\pi/2$?

5. What happens to $\cos\theta$ and $\sin\theta$ as θ increases from 0 to $\pi/2$?

6. Now think about what happens to x and y as θ varies from $-\pi$ to π. When are they positive and when are they negative?

7. Use the previous part to determine when $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive and negative on the interval $(-\pi, \pi]$.

8. Let c be a number in the interval $(-a, a)$. What can you say about the two values of θ such that $x = c$ and the two values of θ such that $y = c$?

9. Now let $d \in (-1, 1)$. Use the previous part to determine the relationship between the two values of θ such that $\sin\theta = d$ and the relationship between the two values of θ such that $\cos\theta = d$.

10. Use the first three parts to prove a well-known trigonometric identity.

Homework for Functions VI—VIII

1. Express the following as the composition of two functions.

   (a) $h : \mathbb{R} \setminus \{2\} \to \mathbb{R},\ h(x) = \dfrac{1}{x-2}$; (b) $h : \mathbb{R} \to \mathbb{R},\ h(x) = \sin^4 2x$;

   (c) $h : \mathbb{R} \to \mathbb{R},\ h(x) = 2\sin^3 x + 5\sin x$; (d) $h : \mathbb{R} \to \mathbb{R},\ h(x) = \dfrac{1}{2+\cos x}$.

2. Mild Challenge Question: Think about one of the parts of the previous question again. It doesn't matter which part. Can you find alternative (correct) answers? How many? Can you find two "trivial" or apparently silly yet still correct answers?

3. Give the complete function specification (domain, codomain and rule) for the compositions $f \circ g$ and $g \circ f$ when:

   (a) $f : [0, \infty) \to \mathbb{R},\ f(x) = \sqrt{x}$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$;

   (b) $f : [2, \infty) \to \mathbb{R},\ f(x) = \sqrt{x-2}$ and $g : (-\infty, 3] \to \mathbb{R},\ g(x) = \sqrt{3-x}$;

   (c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{x^2}{1+x^2}$ and $g : (0, \infty) \to \mathbb{R},\ g(x) = \ln x$.

   Are $f \circ g$ and $g \circ f$ equal in any of the examples?

4. Find two functions f and g such that $f \circ g = g \circ f$.

5. (a) Suppose f is an odd function and $0 \in \operatorname{dom}(f)$. What can you say about $f(0)$?

   (b) What can you say about a function that is both even and odd?

6. Challenge Question: Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function and $g : \mathbb{R} \to \mathbb{R}$ be an even function. Determine (i.e. prove) whether for all such f and g each of the following is either even or odd.

   (a) fg; (b) $f + g$.

7. (a) Sketch the graphs of sin, cos and tan. Then read through the last exercise of the session Functions VIII and check that your graphs show all the properties of sin, cos and tan established in the last exercise of the session.

   (b) Without using a calculator find $\sin x$, $\cos x$ and $\tan x$ for all

   $$x \in \{-\pi, -5\pi/6, -2\pi/3, -\pi/2, -\pi/3, \pi/6, 0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6, \pi\}.$$

8. Find all real numbers that satisfy the following equations.

   (a) $2\cos x + 1 = 0$; (b) $\cos x = -1/\sqrt{2}$; (c) $|\tan x| = 1$;

   (d) $2\cos^2 x - 1 = 0$; (e) $3 - 3\sin x - \cos^2 x = 0$.