MA Core worksheet

Session 19–20: Functions IV–V

Reference: Stewart Chapter 1, Pages 10–18, 37–41.

On Thursday we saw some of the key concepts behind functions. These will keep coming up all through your course. About a third of modules are essentially about functions. In all the other modules, functions will appear at some point. Because this material is so crucial, you are going to spend today consolidating the material you studied on Thursday by doing lots of exercises.

There are lots and lots of exercises on this sheet and probably far too many to finish in the session unless you are extremely fast. Your class group leader will decide which ones to work on, based on homework and what people are finding difficult. They will then set some others for homework.

Revisiting Piecewise Functions and the Absolute Value Function

Exercise. Sketch the following functions:

(a) $f(x) = |x - 2|$; (b) $g(x) = |2 - x|$.

Exercise. A mathematician follows a very boring and unhealthy diet. Up until the 15th day of each month he always eats chicken. After that and until the end of the month he always eats fish. Up until the 10th day of each month he prepares his food as a curry. After that and up until the end of the month he bakes his food.

Reword the description so that there is just one sentence describing exactly what the mathematician eats, i.e. "on days ??? of the month he eats ??? in a ??? style, on days ??? etc."

Now give precise definitions of three functions f, g, h so that $f(n)$ tells us what species of animal he eats on day n of each month, $g(n)$ tells us the style of food he eats on day n of each month and $h(n)$ tells us both the species and the style of food he eats on day n of each month. These functions should have piecewise definitions. Assume, for simplicity, that every month has 30 days.

Exercise. Let

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} a & \text{if } x \leq 15, \\ b & \text{if } x > 15 \end{cases} \quad \text{and} \quad g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} c & \text{if } x \leq 10, \\ d & \text{if } x > 10. \end{cases}$$

Determine a piecewise expression for $f + g$.

Exercise. Let

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} x & \text{if } x \leq 1, \\ x - 1 & \text{if } x > 1 \end{cases} \quad \text{and} \quad g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} \sin x & \text{if } x > \pi, \\ \cos x & \text{if } x \leq \pi. \end{cases}$$

Determine a piecewise expression for $f + g$.

Exercise. Express the following functions using a piecewise definition:

(a) $|x-15|+|x-10|$; (b) $2|x-2|-3|2x-7|$; (c) $|x-1|+|x-2|+|x-3|$;
(d) $|9-x^2|$; (e) $|4-x|+x-4$.

Revisiting Natural Domains and Inequalities

Exercise. Find the natural domains of the following function rules:

(a) $\sqrt{\dfrac{1+2x}{2-x}-1}$;

(b) $\dfrac{1}{\sqrt{x-2}}+\dfrac{1}{\sqrt{4-x}}$;

(c) $\dfrac{1}{\sqrt{x-5}}+\dfrac{1}{\sqrt{4-x}}$;

(d) $\dfrac{1}{(x-3/2)\sqrt{\dfrac{1+2x}{2-x}-1}}$;

(e) $\dfrac{x-1}{x^2-5x+4}$.

Slightly Altering a Function: What Happens to its Graph, Natural Domain and Range?

The object of these exercises is to think about what happens to a function when we make some small changes to its definition. At the same time there is the opportunity to practice finding natural domains and graphs and thinking about the graphs of functions.

Exercise.

1. The point of this exercise is to think about the natural domain, range and graph of $f(x) = \sqrt{x} + a$ where a is some unknown constant.

   To begin with, what are the natural domain and range of $\sqrt{x}$? Sketch the graph of $\sqrt{x}$.

   Now find the natural domain and range of $f(x) = \sqrt{x} + a$ and sketch its graph. If necessary, experiment with different values of a until you understand what is happening.

2. Repeat the previous part but this time with $g(x) = 1/x + a$.

3. What are the natural domain, range and graph of $h(x) = \dfrac{1 + ax}{x}$?

4. Explain what happens to the natural domain, range and graph of a general, arbitrary function when a constant is added to it.

Exercise.

1. The aim of this exercise is to think about the natural domain, range and graph of $f(x) = \sqrt{x-b}$ where b is some unknown constant.

   Find the natural domain and range of $f(x) = \sqrt{x-b}$ and sketch its graph. If necessary, experiment with different values of b until you understand what is happening.

2. Repeat the previous part but this time with $g(x) = 1/(x-b)$.

3. Suppose we let $f_0(x) = \sqrt{x}$ and $g_0(x) = 1/x$. How do you express f in terms of f_0 and g in terms of g_0?

4. Explain what happens to the natural domain, range and graph of a general, arbitrary function when we replace the variable x by $x - b$ for some constant b.

MA Core worksheet

Homework for Sessions 19–20

1. Finish any exercises from this session or from the last homework, as directed by your group leader.

2. Find the natural domain of the following functions and then sketch them:

   (a) $f(x) = \dfrac{1-t^2}{1-t}$; (b) $g(x) = |3x+1|$; (c) $k(x) = \dfrac{|x|}{x^2}$.

3. Mild Challenge Question: using your answers to some of the exercises, find the natural domain, range and graphs of:

   (a) $f(x) = \sqrt{x-b}+a$; (b) $g(x) = \dfrac{1}{x-b}+a$.

4. Challenge Question: find the natural domain and range of the rule $h(x) = \dfrac{1}{\sqrt{x-b}+a}$.