Session 16: Functions I

Reference: Stewart Chapter 1, Pages 10–16.

The main goal of this session is to give an introduction to functions. When the core material splits into distinct strands in a few weeks time, one of the strands will be calculus, which is essentially about studying functions.

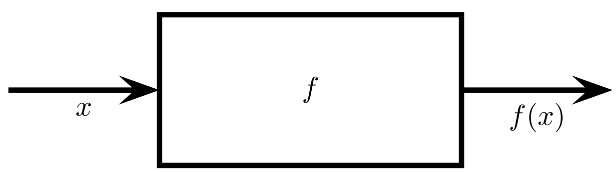
Example. The perimeter p of a circle with radius r is given by $p = 2\pi r$. We say that p is a function of r.

Definition. A function consists of four parts:

1. A name.
2. A set called the domain.
3. A set called the codomain.
4. A rule which assigns a unique element of the codomain to each element in the domain.

Names of functions could be f, g, sin, ln, $f + g$, $f \circ g$ or fred. Notice that the name of the function does not include a variable, that is, the name of the function is f not $f(x)$. Sometimes we may be a bit sloppy with this convention.

The domain is the set of all possible inputs, the codomain is a set to which all the possible outputs belong.



One way to picture a function is as a machine which takes in a number x and spits out another number $f(x)$.

We use the notation $f : A \to B$ to denote a function with domain A and codomain B.

However for almost every function this year, the codomain is just $\mathbb{R}$. So if ever we want to decide the codomain of a function, the answer is $\mathbb{R}$.

Here are two simple examples.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

Question. What is an important fact about $f(x)$ in the previous example?

We always have $f(x) \geq 0$ so not every member of the codomain needs to appear as a value taken by $f(x)$.

Example. Let $g : \{1, 2, 3\} \to \mathbb{R}$ be defined by $g(x) = x$.

In this example only three elements of the codomain are hit by g. But the codomain contains infinitely many elements.

Question. What is $g(4)$ in the previous example?

In general, the codomain of a function must contain every element taken by the function, but not every element of the codomain needs to be hit by the function.

Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 3x + 1$. Find:

(a) $f(2)$; (b) $f(x) - 1$; (c) $f(x - 1)$; (d) $f(1/x)$; (e) $f(t)$; (f) $f(x + h)$.

In order to define a function properly, the definition must include a domain and codomain. Sometimes, particularly when working with common functions, we omit the domain and codomain.

In this case we may need to find sensible choices for the domain and codomain.

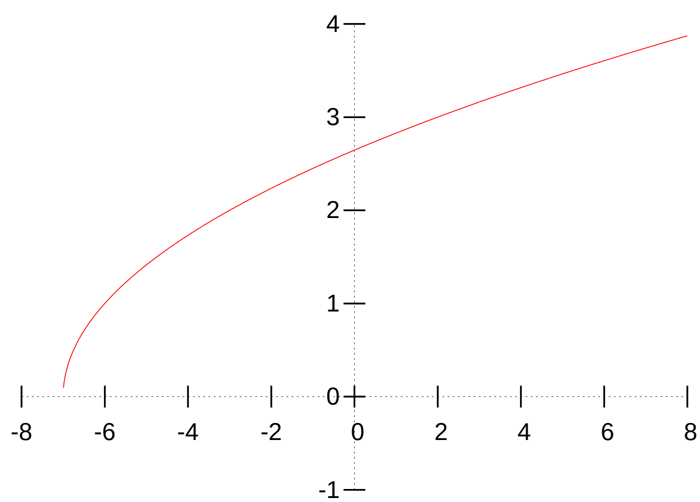
Question. Looking back at what we said earlier, what should we take the codomain to be?

When the domain is missing we take the domain to be the largest subset of $\mathbb{R}$ for which the rule makes sense.

Definition. The natural domain of a function rule f is the largest subset of $\mathbb{R}$ to which the rule may be applied and still make sense.

Example. Find the natural domain and codomain of $f(x) = \sqrt{x + 7}$.

The function $\sqrt{y}$ is only defined if $y \geq 0$, so here we need $x + 7 \geq 0$ or equivalently $x \geq -7$. Hence the natural domain is $[-7, \infty)$. The codomain is $\mathbb{R}$.

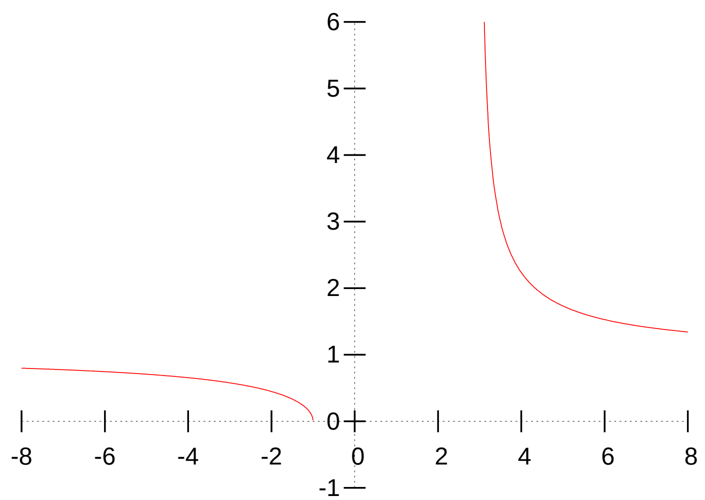


Exercise. Find the natural domains of:

(a) $f(x) = \dfrac{1}{\sqrt{2-x}}$; (b) $g(x) = \dfrac{1}{x(x-1)}$; (c) $h(x) = \dfrac{x}{x(x-1)}$;

(d) $k(x) = \dfrac{1}{\cos x}$; (e) $l(x) = \dfrac{1}{\sqrt{x}-2}$; (f) $m(x) = \sqrt{\dfrac{2x-2}{x-3} - 1}$.

Here is the graph of m.



Summary. We have seen:

1. How the basic components of a function are defined.
2. How to find the natural domain of a function.

Session 17: Functions II

Reference: Stewart Chapter 1, Pages 10–16, 41.

In this session we will look at the range of a function and then see how we can combine functions to get new functions.

Definition. The range of a function is the set

$$\{f(x) : x \text{ is in the domain of } f\}.$$

So the range is the set of all values that $f(x)$ can take as x varies through the domain.

Question. What is the relationship of the range and the codomain? Are they equal?

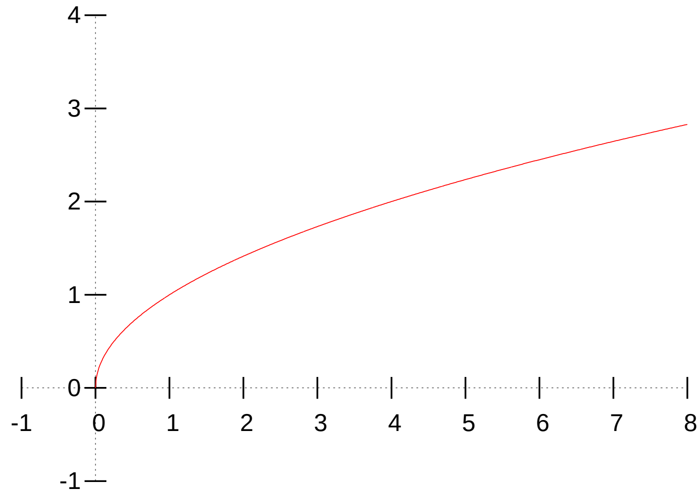
Finding the range may be complicated. It is a good idea to sketch the graph of the function. Sometimes the range will become obvious by sketching the graph. In more complicated situations there are often two stages to finding the range: determining which numbers cannot be in the range and making sure that all the other numbers are in the range.

Example. Find the range of $k : [0, \infty) \to \mathbb{R}$ given by $k(x) = \sqrt{x}$.

We know that $\sqrt{x} \geq 0$ so no negative numbers appear in the range.

We now check that every positive number appears in the range. In general, the idea is to put $y = k(x)$ and try to solve for x in terms of y. If we get a sensible answer for x then $k(x) = y$ and so y is in the range.

In this example put $y = \sqrt{x}$, then $x = y^2$. So if $y \geq 0$ then $\sqrt{y^2} = y$ and hence the range is $[0, \infty)$.

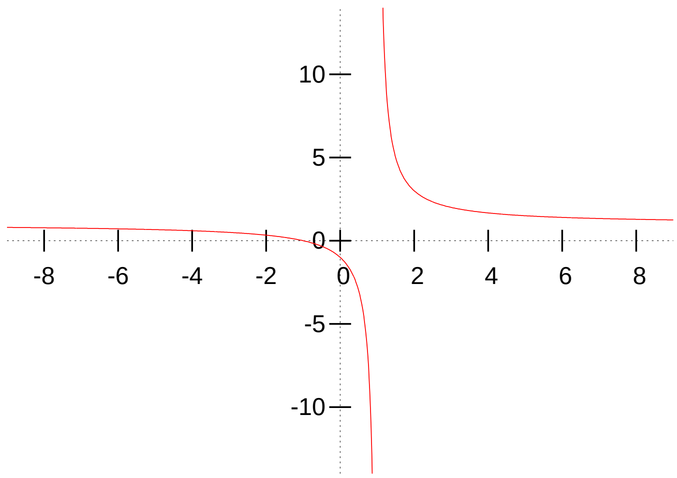


Exercise. Find the range of:

(a) $f : [1, \infty) \to \mathbb{R},\ f(x) = 2 + \sqrt{x - 1}$; (b) $g : [-2, 1] \to \mathbb{R},\ g(x) = x^2$;

(c) $h : \mathbb{R} \setminus \{1\} \to \mathbb{R},\ h(x) = \dfrac{x+1}{x-1}$.

Here is the graph of h.



Combining functions by adding, subtracting, multiplying or dividing them allows us to build up a vast array of functions from the basic ones.

We will use $\mathrm{dom}(f)$ to denote the domain of a function f.

Definition. Suppose that f and g are two functions with codomain $\mathbb{R}$. Then:

1. $f+g$ is the function with domain $\mathrm{dom}(f)\cap\mathrm{dom}(g)$, codomain $\mathbb{R}$ and $(f+g)(x) = f(x)+g(x)$.
2. $f-g$ is the function with domain $\mathrm{dom}(f)\cap\mathrm{dom}(g)$, codomain $\mathbb{R}$ and $(f-g)(x) = f(x)-g(x)$.
3. fg is the function with domain $\mathrm{dom}(f)\cap\mathrm{dom}(g)$, codomain $\mathbb{R}$ and $(fg)(x) = f(x)g(x)$.
4. f/g is the function with domain $(\mathrm{dom}(f)\cap\mathrm{dom}(g))\setminus\{x\in\mathrm{dom}(g) : g(x)=0\}$, codomain $\mathbb{R}$ and $(f/g)(x) = \dfrac{f(x)}{g(x)}$.

What we have done is to define new functions with names $f+g$, $f-g$, fg and f/g. Their domains and codomains are defined precisely. Their rules should be what you would expect.

Question. Why is the definition of the domains of each of the new functions sensible?

Example. Find the domain, codomain and rule of $f+g$ and f/g when

$$f : [-4, 9] \to \mathbb{R},\ f(x) = x+1 \quad \text{and} \quad g : [2,\infty) \to \mathbb{R},\ g(x) = \sqrt{x-2}.$$

The rule for $f+g$ is defined to be

$$f(x) + g(x) = x + 1 + \sqrt{x-2}.$$

The domain of $f+g$ is defined to be $\mathrm{dom}\, f \cap \mathrm{dom}\, g = [-4, 9] \cap [2,\infty) = [2, 9]$.

The codomain of $f+g$ is $\mathbb{R}$.

The rule for f/g is defined to be

$$\frac{f(x)}{g(x)} = \frac{x+1}{\sqrt{x-2}}.$$

The domain of f/g is defined to be $(\mathrm{dom}\, f \cap \mathrm{dom}\, g) \setminus \{x \in \mathrm{dom}\, g : g(x) = 0\}$. In other words it is the same as $\mathrm{dom}(f+g)$ except that we must remove the values where $g(x) = 0$.

$g(x) = 0$ implies $\sqrt{x-2} = 0$ which occurs when $x = 2$.

So $\mathrm{dom}(f/g) = [2, 9] \setminus \{2\} = (2, 9]$.

The codomain of f/g is $\mathbb{R}$.

Exercise. Find the domain, codomain and rule of $f+g$ and f/g when

$$f : \mathbb{R}\setminus\{3\} \to \mathbb{R},\ f(x) = \frac{1}{x-3} \quad \text{and} \quad g : (1,\infty)\setminus\{3\} \to \mathbb{R},\ g(x) = \frac{1}{\sqrt{x-1}(x-3)}.$$

Exercise. Find the domain, codomain and rule of $f + g$, fg and f/g when

$$f : [0, \infty) \to \mathbb{R},\ f(x) = \sqrt{x} \quad \text{and} \quad g : [0, \infty) \to \mathbb{R},\ g(x) = \sqrt{x}.$$

Summary. We have seen:

1. How the range of a function is defined and its relationship with the codomain.
2. How to find the range of a function.
3. How simple combinations of functions are defined and the importance of the precise definition of their domain.

Session 18: Functions III

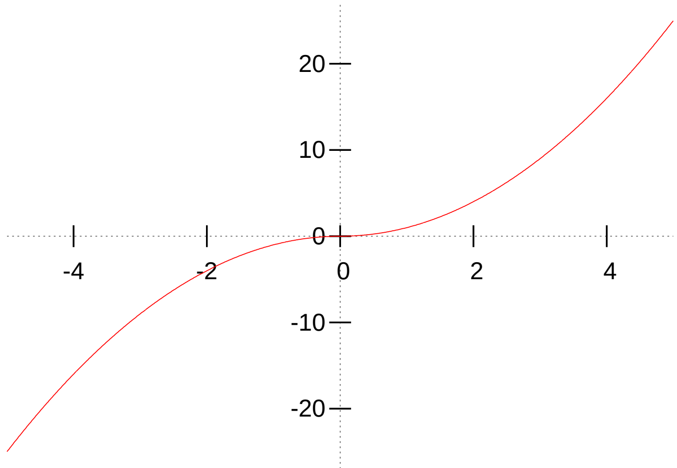
Reference: Stewart Chapter 1, Pages 17–18.

In this session we will look at what it means for a function to have a piecewise definition and then see how to express functions containing an absolute value in this way.

Definition. A piecewise definition of a function is one where the rule comes in several pieces and the one that must be applied depends on the value of the variable.

Example.

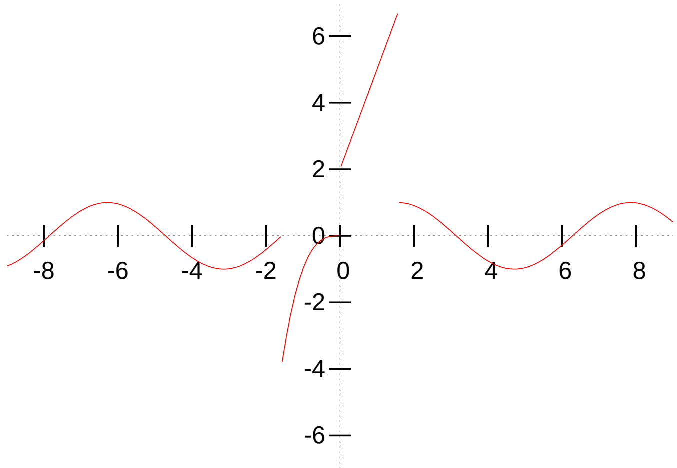
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$



A piecewise function can have many pieces.

Example.

$$f(x) = \begin{cases} \sin x & \text{if } x \geq \pi/2, \\ 3x+2 & \text{if } 0 \leq x < \pi/2, \\ x^3 & \text{if } -\pi/2 < x < 0, \\ \cos x & \text{if } x \leq -\pi/2. \end{cases}$$



Notice that each possible value of x appears in exactly one of the pieces.

Sometimes a piecewise function is hard to sketch and may appear quite exotic.

Example.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Piecewise definitions often appear when we are dealing with functions involving absolute values. Remember from Session 12 that

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Remember also that the graph of $|f(x)|$ is obtained by taking the graph of $f(x)$ and reflecting the parts where $f(x)$ is negative in the x-axis.

The most important point is:

The rule that applies depends on the sign of $f(x)$ not on the sign of x.

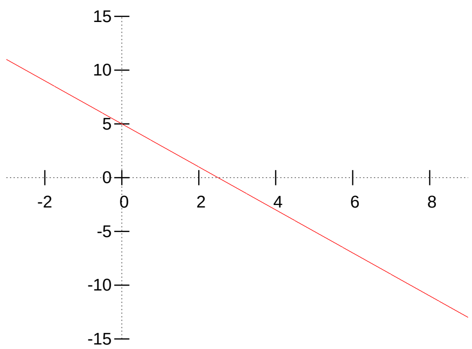
Example. Rewrite the function $f(x) = |5 - 2x|$ without using modulus signs.

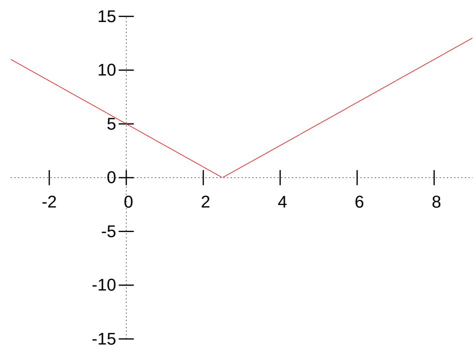
Apply the definition of $|f(x)|$ to $|5 - 2x|$.

$$f(x) = \begin{cases} 5 - 2x & \text{if } 5 - 2x \geq 0, \\ 2x - 5 & \text{if } 5 - 2x < 0 \end{cases} = \begin{cases} 5 - 2x & \text{if } x \leq 5/2, \\ 2x - 5 & \text{if } x > 5/2. \end{cases}$$

In the last step we have simplified / solved $5 - 2x \geq 0$ and $5 - 2x < 0$.

The graphs of $5 - 2x$ and $|5 - 2x|$ are





Exercise. Rewrite the following functions using a piecewise definition to remove the modulus sign. In each case try to find the answer without sketching the function, then sketch the function and think again about your answer.

(a) $f(x) = |7 - x|$; (b) $g(x) = |2x - 3|$; (c) $h(x) = |2x - 3| + x + 1$;

(d) $k(x) = |x^2 - 3|$; (e) $l(x) = |x - 1| + |x + 1|$; (f) $m(x) = |2 - x| + |2x - 3|$.

Homework for Sessions 16–18

1. Find the natural domains and ranges of the following functions:

   (a) $g(x) = 4 - x^2$; (b) $k(x) = \dfrac{1}{3 - \sqrt{x}}$; (c) $f(x) = \sqrt{4 - x^2}$.

2. Express the following functions in piecewise form without using absolute values:

   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = |x + 5|$;
   (b) $g : \mathbb{R} \to \mathbb{R},\ g(x) = |x^2 - 4|$;
   (c) $h : \mathbb{R} \to \mathbb{R},\ h(x) = |x + 1| + x + 1$;
   (d) $k : \mathbb{R} \to \mathbb{R},\ k(x) = 3 + |2x - 5|$;
   (e) $l : \mathbb{R} \to \mathbb{R},\ l(x) = |x| + |x - 1|$;
   (f) $m : \mathbb{R} \to \mathbb{R},\ m(x) = 3|x - 2| - |x + 1|$.

3. Find the domain and rule for the functions fg and f/g when f and g are as follows:

   (a) $f : [-2, \infty) \to \mathbb{R},\ f(x) = \dfrac{x}{1 + x^2}$ and $g : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ g(x) = \dfrac{1}{x}$.

   (b) $f : [2, \infty) \to \mathbb{R},\ f(x) = \sqrt{x - 2}$ and $g : [3, \infty) \to \mathbb{R},\ g(x) = \sqrt{x - 3}$.

4. Challenge Question: find the natural domain and range of the rule $h(x) = \dfrac{3}{1 - \cos x}$.

5. Challenge Question: let m be the function from question 2f. Determine $\displaystyle\int_{-2}^{3} m(x)\,\mathrm{d}x$.