MA1972
Exam Solutions and Marking Scheme
Summer 2008

1. TYPE OF DEGREE: BSc.

2. SESSION: Summer 2008.

3. MODULE CODE: MA1972.

4. MODULE TITLE: Discrete Mathematics, Probability & Statistics.

5. TIME ALLOWED: 2 hours plus 5 minutes reading.

6. a. Answer all questions from Sections A

Answer two questions from section B. If more than 2 questions from section B are answered then the best two answers will be counted.

Section A carries 50% of the marks for the paper.

All questions in section B are worth equal marks.

40% needed in both sections??

7. ADDITIONAL INFORMATION: Neave statistical tables are provided.

Sue: anything else?

Section A

A1 $|\{x \in \mathbb{Z}\colon 0 < \sqrt{x} \leqslant 3\}| = |\{1, 2, 3, 4, 5, 6, 7, 8, 9\}| = 9.$ [3 marks]

A2 The statement $1 + \sum_{m=1}^{n} 2^m = 2^{n+1} - 1$ is obviously true when $n = 1$ and if it is true for $n = k$ then it is also true for $n = k + 1$ because, [2 marks]

$$1 + \sum_{m=1}^{k+1} 2^m = 1 + \sum_{m=1}^{k} 2^m + 2^{k+1} = 2 \times 2^{k+1} - 1 = 2^{k+2} - 1.$$

Hence it is true for all $n \in \mathbb{N}$. [3 marks]

A3 We can take the sample space as $\Omega = \{OO, OE, EO, EE\}$ so that $A = \{OE, OO\}$, $B = \{OE, EE\}$ and $C = \{OO, EE\}$. Since all outcomes are equally likely we have $P(A) = |A|/|\Omega| = 1/2$. Similarly, $P(B) = P(C) = 1/2$. Now, [3 marks]

$$\begin{aligned} P(A \cap B) &= P(\{OE\}) = \frac{1}{4} = P(A)P(B), \\ P(B \cap C) &= P(\{EE\}) = \frac{1}{4} = P(B)P(C), \\ P(A \cap C) &= P(\{OO\}) = \frac{1}{4} = P(A)P(C), \end{aligned}$$

which shows that A, B and C are pairwise independent. On the other hand, [5 marks]

$$P(A \cap B \cap C) = P(\varnothing) = 0 \neq P(A)P(B)P(C),$$

and so they are not collectively independent. [2 marks]

A4 If $X \sim B(n, p)$ for $n = 5$ and $p = 1/4$ then: $E(X) = np = 5/4$; $\mathrm{Var}(X) = np(1 - p) = 15/16$; $E(X^2) = \mathrm{Var}(X) + E(X)^2 = 15/16 + 25/16 = 5/2$. Also, [3 marks]

$$P(X = 3) = \binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{45}{512} \approx 8.8\%$$

and, [3 marks]

$$\begin{aligned} P(X \geqslant 4) &= P(X = 4) + P(X = 5), \\ &= \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 + \binom{5}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0, \\ &= \frac{5 \times 3}{4^5} + \frac{1}{4^5} = \frac{16}{4^5} = \frac{1}{64} \approx 1.56\%. \end{aligned}$$

[1 mark]

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Section B

B1 a. **Either:** $\Omega = \{BB, BR\}$ and $E = \{BB\}$ giving $P(E) = \frac{1}{2}$.
Or:

$$P(\{BB\} \,|\, \{BB, BR\}) = \frac{P(\{BB\} \cap \{BB, BR\})}{P(\{BB\} \cup \{BR\})}$$
$$= \frac{P(\{BB\})}{P(\{BB\}) + P(\{BR\})} = \frac{1/3}{1/3 + 1/3} = \frac{1}{2}.$$

[3 marks]

b. Define the event $A = \{\text{student achieves a grade } A\}$, with B, C and F similarly defined.
These events are disjoint and we are given [2 marks]

$$P(A) = 0.2, \qquad P(A \cup B) = 0.38, \qquad P(A \cup B \cup C) = P(F^c) = 0.65.$$

Hence,

(i) $P(F) = 1 - P(F^c) = 1 - 0.65 = 35\%$. [1 mark]

(ii) $P(B) = P(A \cup B) - P(A) = 0.38 - 0.2 = 18\%$. [2 marks]

(iii) $P(C) = P(A \cup B \cup C) - P(A \cup B) = 0.65 - 0.38 = 27\%$. [2 marks]

c. Let X be the event that a student attended a revision class. We have $P(A|X) = 0.3$ and $P(X) = 0.45$.
Hence,

(i) $0.2 = P(A) = P(A|X)P(X) + P(A|X^c)P(X^c) = 0.3 \times 0.45 + P(A|X^c) \times 0.55$ and so,

$$P(A|X^c) = \frac{0.2 - 0.3 \times 0.45}{0.55} = 11.818\ldots\%$$

[4 marks]

(ii) $P(X|A) = \dfrac{P(A|X)P(X)}{P(A)} = \dfrac{0.3 \times 0.45}{0.2} = 67.5\%$. [3 marks]

d. Let E_m be the event of a pass on the $m-$th attempt. Then $P(E_m) = 4/10$ and $P(E_m^c) = 6/10$ for each $m = 1, 2, \ldots$. [1 mark]
Let $X \in \mathbb{N}$ be the number of attempts until a pass is obtained then, with $p = 4/10$,

$$P(X = m) = p(1-p)^{m-1} \qquad \text{and so} \qquad X \sim \text{Geom}(p).$$

Hence, [2 marks]

(i) ... on the third try:

$$P(X = 3) = \frac{4}{10}\frac{6^2}{10^2} = \frac{144}{1000} = \frac{18}{125} \approx 14.5\%.$$

[2 marks]

(ii) ... before the third try:

$$P(X < 3) = P(X = 1) + P(X = 2) = \frac{4}{10}\left(1 + \frac{6}{10}\right) = \frac{16}{25} = 64\%.$$

The average number of attempts is $E(X) = {}^{1}/_{p} = 2.5$. [3 marks]

B2 a. Choose 8 leaving 22, then 15 leaving 7:

$$\binom{30}{8}\binom{22}{15} = \frac{30!}{8!\ 22!}\frac{22!}{15!\ 7!} = \frac{30!}{8!\ 15!\ 7!} \approx 10^{12}.$$

[4 marks]

b. Choose two from each room and multiply...

$$\binom{15}{2}\binom{8}{2}\binom{7}{2} = 61,740.$$

[3 marks]

c. There are $26+26+10 = 62$ characters giving $62^6 \approx 5.68 \times 10^{10}$ passwords.

[3 marks]

Four distinct letters followed by two distinct numbers can be chosen in

$$52 \times 51 \times 50 \times 49 \times 10 \times 9 = 584,766,000$$

ways. The two numbers can be inserted into the password in any one of $\binom{4}{2} = 6$ ways giving,

$$6 \times 584,766,000 = 3,508,596,000$$

passwords. [7 marks]

d. Choose the first pair in $\binom{30}{2}$ ways, the second pair in $\binom{28}{2}$ ways and so on and then multiply:

$$\begin{aligned}&\binom{30}{2}\binom{28}{2}\binom{26}{2}\cdots\binom{4}{2}\binom{2}{2}\\ &= \frac{30!}{2!\ 28!} \times \frac{28!}{2!\ 26!} \times \frac{26!}{2!\ 24!} \times \cdots \frac{4!}{2!\ 2!} \times \frac{2!}{2!\ 0!},\\ &= \frac{30!}{(2!)^{15}} \approx 8.1 \times 10^{27}\end{aligned}$$

different pairs. [4 marks]

However, these same pairs can arise in 15! ways so there are only

$$\frac{30!}{15!\ (2!)^{15}} \approx 6.19 \times 10^{15}$$

distinct ways. [2 marks]

There are $15! \approx 1.31 \times 10^{12}$ ways the pairs can take their places on the coach. (The order in which each pair sits is not needed, but credit a non-full score if a sensible attempt is made.) [2 marks]