Marking scheme and solutions for MA1972 (2007)

1. TYPE OF DEGREE: BSc.

2. SESSION: May 2007.

3. MODULE CODE: MA1972.

4. MODULE TITLE: Discrete Mathematics, Probability & Statistics.

5. TIME ALLOWED: 2 hours plus 5 minutes reading.

6. a. Answer all questions from Sections A

Answer two questions from section B. If more than 2 questions from section B are answered then the best two answers will be counted.

Section A carries 50% of the marks for the paper.

All questions in section B are worth equal marks.

7. ADDITIONAL INFORMATION: Neave statistical tables are provided.

Section A

A1 $n=1$ gives $\frac{1}{3}(7^1-1)=6/3=2\in\mathbb{N}$.

Assume $p=\frac{1}{3}(7^n-1)\in\mathbb{N}$ for some $n\in\mathbb{N}$ then,

$$\frac{1}{3}(7^{n+1}-1)=\frac{1}{3}(7^{n+1}-7^n)+p=2\times 7^n+p\in\mathbb{N}.$$

Hence $\frac{1}{3}(7^n-1)\in\mathbb{N}$ for every $n\in\mathbb{N}$. [4 marks]

A2 a. $|\{2,3,4,8,9\}|=5$ [1 mark]

b. $|\mathcal{P}(A)|=2^7$ and $|\mathcal{P}(B)|=2^9$.
Each $X\in\mathcal{P}(A)$ can be paired with all 2^9 elements of $\mathcal{P}(B)$.
Hence $|\mathcal{P}(A)\times\mathcal{P}(B)|=2^{7+9}=2^{16}=65536$. [3 marks]

c. Let $X\in\mathcal{P}(A)\cup\mathcal{P}(B)$ be arbitrary.
Either $X\in\mathcal{P}(A)$ or $X\in\mathcal{P}(B)$ or both.
If $X\in\mathcal{P}(A)$ then $X\subseteq A$ and $X\subseteq A\cup B$.
Similarly $X\subseteq A\cup B$ if $X\in\mathcal{P}(B)$. [4 marks]

A3 a. $26^6\times 10^4\approx 3.1\times 10^{12}$. [2 marks]

b. $52^6\times 10^4\approx 1.98\times 10^{14}$. [1 mark]

A4

$$P(A)=P(A\cap B)+P(A\cup B)-P(B)=\frac{1}{10}+\frac{4}{5}-\frac{1}{2}=\frac{2}{5}$$

[2 marks]

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)}=\frac{1/10}{1/2}=\frac{1}{5}.$$

[2 marks]

$$P(B\,|\,A^c)=\frac{P(A^c\cap B)}{P(A^c)}=\frac{P(B)-P(A\cap B)}{1-P(A)}=\frac{1/2-1/10}{1-2/5}=\frac{2}{3}.$$

[4 marks]

A and B are independent if $P(A\cap B)=P(A)P(B)$.

Since $1/10\neq 1/5$, A and B are not independent. [2 marks]

Section B

B1 Define the events:

$$\begin{aligned} S &= \{\text{served by Sam}\}, \\ M &= \{\text{served by Mo}\}, \\ Z &= \{\text{customer does not get a voucher}\}. \end{aligned}$$

a. (i) $P(Z^c \mid M) = 11/12 \approx 0.917.$ [2 marks]

(ii) Either $P(Z^c) = 1 - P(Z) = 14/15 \approx 0.93$ because,

$$P(Z) = P(Z \mid S)P(S) + P(Z \mid M)P(M) = \frac{1}{20} \times \frac{1}{2} + \frac{1}{12} \times \frac{1}{2} = \frac{1}{15},$$

or

$$P(Z^c) = P(Z^c \mid S)P(S) + P(Z^c \mid M)P(M) = \frac{1}{2}\left(\frac{19}{20} + \frac{11}{12}\right) = \frac{14}{15}.$$

[5 marks]

(iii)

$$P(M \mid Z) = \frac{P(M \cap Z)}{P(Z)} = \frac{P(M \cap Z)}{P(M)} \frac{P(M)}{P(Z)} = \frac{P(Z \mid M)P(M)}{P(Z)}.$$

Hence,

$$P(M \mid Z) = \frac{1/12 \times 1/2}{1/15} = \frac{5}{8}.$$

[6 marks]

b. We have $X \sim B(50, 0.1)$ so,

$$\begin{aligned} P(X > 2) &= 1 - P(X \leqslant 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2), \\ &= 1 - \binom{50}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{50} - \binom{50}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{49} \\ &\quad - \binom{50}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{48}, \\ &= 1 - \left(\frac{9}{10}\right)^{48}\left(\frac{1756}{100}\right) \approx 0.888. \end{aligned}$$

[6 marks]

$\mathbb{E}(X) = np = 50 \times 0.1 = 5.$ [1 mark]

$\mathrm{Var}(X) = np(1-p) = 5 \times 0.9 = 4.5.$ [1 mark]

c. If $Y = m$ then the first $m - 1$ customers do not order a Gigabite burger. Hence, with $p = 0.1$,

$$P(Y = m) = (1 - p)^{m-1}p.$$

Therefore Y follows the geometric distribution with parameter $p = 0.1$. [4 marks]

B2 a. There are $\binom{30}{4} = 27,405$ possible plays.

E_4: $P(E_4) = \binom{30}{4}^{-1} \approx 0.000036489.$ [1 mark]

$E_3^\star$: There are $\binom{4}{3}$ ways to match any three main balls and the fourth ball can only be the bonus ball. Hence,

$$P(E_3^\star) = \binom{4}{3}\binom{30}{4}^{-1} = \frac{4}{27,405} \approx 0.000146.$$

[2 marks]

E_3: Any three main balls can be matched in $\binom{4}{3}$ ways and there are 25 possibilities for the remaining ball (excluding the bonus ball and the other main ball). Hence,

$$P(E_3) = 25 \times \binom{4}{3}\binom{30}{4}^{-1} = \frac{100}{27,405} \approx 0.00365.$$

[2 marks]

b. There are $\binom{52}{3} = 22,100$ possible hands.

(i) P('a triple of threes') $= \binom{4}{3}\binom{52}{3}^{-1} = \frac{4}{22,100} = \frac{1}{5525} \approx 0.000181.$ [3 marks]

(ii) P('a triple of any value *except* three') is given by,

$$12 \times \binom{4}{3}\binom{52}{3}^{-1} = \frac{48}{22,100} \approx 0.00217.$$

[4 marks]

(iii) There are 13 possibilities for any one of the three and for each of the two others there are three ways to make a pair. The third (non-paired) card can be any one of 48. Hence, P('a pair') is given by,

$$\frac{13 \times 2 \times 3 \times 48}{22,100} = \frac{3744}{22,100} = \frac{72}{425} \approx 0.1694.$$

[4 marks]

c. We are told that $P(X = 6) = P(X = 1) = 2P(X = 4)$ and $P(X = 3) = P(X = 4) = 2P(X = 2)$.

(i) Let $p = P(X = 2) = P(X = 5)$ then $P(X = 3) = P(X = 4) = 2p$ and $P(X = 1) = P(X = 6) = 4p$. Since these probabilities sum to unity we get $14p = 1$ and so $p = 1/14 \approx 0.07143$. [4 marks]

(ii) $\mathbb{E}(X) = \sum_{x=1}^{6} xP(X = x) = (4 + 2 + 6 + 8 + 5 + 24)p = 49/14 = 3.5.$

[2 marks]

$\mathbb{E}(X^2) = \sum_{x=1}^{6} x^2P(X = x) = (4+4+18+32+25+144)p = 227/14 = 16.2143\ldots.$

Hence $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 227/14 - 49^2/14^2 \approx 3.9643.$ [3 marks]