1. TYPE OF DEGREE: BSc.

2. SESSION: May 2007.

3. MODULE CODE: MA1972.

4. MODULE TITLE: Discrete Mathematics, Probability & Statistics.

5. TIME ALLOWED: 2 hours plus 5 minutes reading.

6. a. Answer all questions from Sections A
Answer two questions from section B. If more than 2 questions from section B are answered then the best two answers will be counted.
Section A carries 50% of the marks for the paper.
All questions in section B are worth equal marks.

7. ADDITIONAL INFORMATION: Neave statistical tables are provided.

Section A

A1 Prove that $\frac{1}{3}\left(7^n - 1\right) \in \mathbb{N}$ for every $n \in \mathbb{N}$. [4 marks]

A2 a. Determine $|\{x \in \mathbb{N}\colon 3 < x^2 \leqslant 16 \text{ or } 8 \leqslant x < 10\}|$. [1 mark]

b. Find $|\mathcal{P}(A) \times \mathcal{P}(B)|$ when $A = \{1, 2, \ldots, 7\}$ and $B = \{1, 2, \ldots, 9\}$. [3 marks]

c. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ for all sets A and B. [4 marks]

A3 The manufacturer's serial number for an mp3 player is to begin with six upper-case characters taken from {A, B, . . . , Z} which are followed by four digits taken from {0,1,2,. . . ,9}.

a. How many distinct serial numbers are possible? [2 marks]

b. Now suppose that the first six characters can be a mixture of both upper- and lower-case. How many distinct serial numbers are there now? [1 mark]

A4 Let A and B be events such that $P(A \cap B) = {}^1/_{10}$, $P(A \cup B) = {}^4/_5$ and $P(B) = {}^1/_2$. Determine

$$P(A), \qquad P(A \,|\, B) \qquad \text{and} \qquad P(B \,|\, A^c).$$

Explain what it means for two events to be independent. Are A and B independent? [10 marks]

A5 A local optician wishes to estimate the average time it takes to give a patient an eye test. During the last few weeks they have recorded the time taken to perform eye tests on a random sample of 150 patients, and found that the mean was 8.7 minutes with a standard deviation of 2.3 minutes.

Calculate the 95% confidence interval for the average time taken to perform eye tests for all patients at this practice. [3 marks]

A6 Let $X_1, X_2, \ldots, X_n$ be independent normally distributed random variables each having mean μ and variance σ^2. State the distribution, the mean and the variance of the random variable

$$T = X_1 + X_2 + \cdots + X_n.$$

[4 marks]

A7 A continuous random variable X takes all values x in the range $0 < x < 1$ with the probability density function (p.d.f)

$$f(x) = \begin{cases} cx^{\alpha-1}, & 0 < x < 1; \\ 0, & \text{elsewhere,} \end{cases}$$

where α and c are positive constants.

a. Show that $c = \alpha$ in order for $f(x)$ to be a valid p.d.f. [2 marks]

b. Show that the mean and variance of X are respectively;

$$\frac{\alpha}{\alpha+1} \qquad \text{and} \qquad \frac{\alpha}{(\alpha+2)(\alpha+1)^2}.$$

[6 marks]

c. Find the probability that a realised value of X will exceed the mean value. [3 marks]

A8 The results from the 2004 Barclays Annual Graduate Survey were entered into SPSS. In order to establish if graduates seeking employment find age has an effect on the attitude of potential employees a χ^2 test was performed. Some of the results are shown in the tables below;

Current employment situation * Agegroup Crosstabulation

			Agegroup					
			18-21	22-25	26-30	31-40	41 and over	Total
Current employment situation	In permanent employment	Count	428	585	60	87	70	1230
		Expected Count	445.6	588.1	62.5	76.0	57.8	1230.0
	In short term employment	Count	146	160	11	9	5	331
		Expected Count	119.9	158.3	16.8	20.4	15.6	331.0
	Seeking employment	Count	89	130	22	17	11	269
		Expected Count	97.5	128.6	13.7	16.6	12.6	269.0
Total		Count	663	875	93	113	86	1830
		Expected Count	663.0	875.0	93.0	113.0	86.0	1830.0

Chi-Square Tests

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-Square	32.311[a]	8	.000
Likelihood Ratio	35.540	8	.000
Linear-by-Linear Association	3.479	1	.062
N of Valid Cases	1830		

a. 0 cells (.0%) have expected count less than 5. The minimum expected count is 12.64.

a. State the null and alternative hypotheses for this test. [2 marks]

b. What are the test results and what conclusions can be drawn about the null hypothesis? [5 marks]

Section B

B1 Sam and Mo each work the same shift as the only two servers in a take-away burger bar. The bar has just launched a new burger called the *Gigabite* and, to boost sales, each customer is given a voucher entitling them to a half-price *Gigabite* burger on their next visit.

a. Sam forgets to pass this voucher to one in every twenty customers and Mo forgets for one in every twelve. Every customer is equally likely to be served by Sam or Mo.

(i) A customer is served by Mo. What is the probability that this customer receives a voucher? [2 marks]

(ii) A customer arrives home with her take-away. What is the probability that she has a voucher? [5 marks]

(iii) What is the probability that a customer was served by Mo given that he has no voucher? [6 marks]

b. One in every ten customers orders a *Gigabite* burger independently of any other customer's choice. Let X denote the number of *Gigabite* burgers sold before the fifty-first customer arrives. Determine:

$$P(X > 2), \qquad \mathbb{E}(X) \qquad \text{and} \qquad \text{Var}(X).$$

[8 marks]

c. Each day, the first customer to order a *Gigabite* is given a free soft drink. Suppose this is the Y-th customer to visit the bar. Determine an expression for $P(Y = m)$. What is the name of the probability distribution followed by Y? [4 marks]

B2 a. In a lottery game there are thirty balls labelled $1, 2, \ldots, 30$. A 'draw' consists of randomly choosing four 'main' balls followed by one 'bonus' ball.

To play the game you select four distinct numbers from $\{1, 2, \ldots, 30\}$ before the draw and you win a prize if any of the following events occur. Calculate the probability of each.

E_4: your choice matches all four main balls. [1 mark]

$E_3^\star$: your choice matches exactly three of the main balls and the bonus ball. [2 marks]

E_3: your choice matches exactly three of the main balls. [2 marks]

b. In the card game *three card brag* you are dealt three cards from a randomly shuffled standard fifty-two card deck. Determine the probability of receiving:

(i) A triple of threes (e.g. $(3\heartsuit, 3\spadesuit, 3\clubsuit)$). [3 marks]

(ii) A triple of any value *except* three (e.g. $(Q\clubsuit, Q\diamondsuit, Q\heartsuit)$). [4 marks]

(iii) A pair (e.g. $(5\heartsuit, 5\spadesuit, 9\diamondsuit)$). [4 marks]

c. A six-sided die is biased such that $P(X = 6) = P(X = 1)$, $P(X = 5) = P(X = 2)$ and $P(X = 4) = P(X = 3)$, where X is the number shown when the die is thrown.

Given that a six is twice as likely as a four and that a three is twice as likely as a two, determine:

(i) The probability mass function for the die. [4 marks]

(ii) $\mathbb{E}(X)$ and $\mathrm{Var}(X)$. [5 marks]

B3 a. At a dairy, milk is sold in '1 litre' cartons. The cartons used have an overall capacity of 1.01 litre. The filling machine dispenses an amount, that varies according to a distribution that is approximately Normal with a standard deviation of 0.008 litres and a mean level that can be set by the operator;

If the operator sets the mean level at 1.005 litres, what proportion of cartons will:

(i) overflow? [3 marks]

(ii) contain less than 0.99 litre? [2 marks]

b. If 16 cartons of milk are randomly selected, what is the probability that the mean contents of the 16 cartons will be less than 1 litre? [3 marks]

c. The time to failure in hours of the weighing machines have an exponential distribution with mean 250 hours.

Calculate the probability that a weighing machine;

(i) does not fail during a working month of 448 hours. [2 marks]

(ii) fails within a working week of 112 hours. [3 marks]

d. Asdco supermarket chain monitor the accuracy of their checkout operators and they have found that 5% of their till receipts throughout all supermarkets in the UK contain an error of some kind.

Recently the customer service department has received several complaints of checkout errors from customers of their Uxbridge branch and an operations analyst from head office was sent to investigate whether the standard of accuracy had deteriorated at that branch.

The analyst carefully checked 400 till receipts against the purchases and found that 32 of them contained an error of some kind.

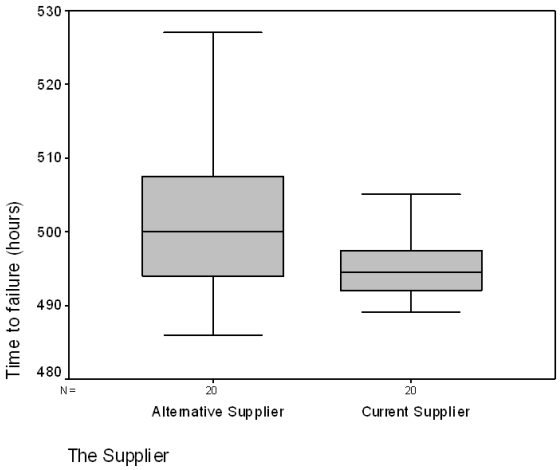
(i) Use an appropriate statistical test to see if the standard of accuracy has deteriorated at the Uxbridge branch. [8 marks]

(ii) How many till receipts should be checked if the proportion of till receipts containing errors is to be estimated to within 1%? [4 marks]

B4 A software company is looking at the time to failure (hours) of the DVDs it uses, and is considering an alternative supplier of DVDs. Random samples of DVDs from the current and new supplier were obtained and the time to failure of each DVD were recorded in SPSS. A summary of results is shown in the table below:

Group Statistics

	The Supplier	N	Mean	Std. Deviation	Std. Error Mean
Time to failure (hours)	Alternative Supplier	20	501.70	10.116	2.262
	Current Supplier	20	495.30	4.485	1.003



a. Using the summary statistics in the table <u>above</u> is there evidence at the 5% significance level of a difference in the variances of the times to failure of the DVDs of the two suppliers? (State hypotheses and all formulae used). [8 marks]

/Continued over page

b. The company has decided that if the mean time to failure for the new supplier is longer than it is for the current supplier, it will switch suppliers.

Independent Samples Test

		Levene's Test for Equality of Variances		t-test for Equality of Means					
								95% Confidence Interval of the Difference	
		F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Lower	Upper
Time to failure (hours)	Equal variances assumed	10.11	.003	2.587	38	.014	6.40	1.391	11.409
	Equal variances not assumed			2.587	26.192	.016	6.40	1.316	11.484

(i) A t-test was perfomed using SPSS and the table above gives the results of two t-test procedures. Explain the difference between these two tests, and specify which procedure applies to this problem. Justify your choice. [5 marks]

(ii) At the 5% level of significance is there evidence that the company should switch supplier? (State hypotheses used). [7 marks]

(iii) Calculate the 95% confidence interval for the difference in mean time to failure between the two suppliers. [5 marks]