1. TYPE OF DEGREE: BSc.

2. SESSION: May 2007.

3. MODULE CODE: MA1920.

4. MODULE TITLE: Linear Algebra.

5. TIME ALLOWED: **TWO hours** plus **five minutes** reading time.

6. a. This paper consists of TWO SECTIONS.
   Answer ALL questions of SECTION A.
   Answer TWO questions out of FOUR from SECTION B. If more than TWO questions from Section B are attempted, only the marks from the best TWO solutions will be counted.
   Section A carries 50% of the total marks for the paper. Each question in Section B carries equal marks. An indication of the marks allocated to each sub-section of a question is shown in square brackets in the right hand margin.

7. ADDITIONAL INFORMATION: STANDARD FORMULAE:
   The matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents an anticlockwise rotation of angle θ about the origin.
   The matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ represents a reflection across the line $y = (\tan\theta)x$.

SECTION A

A1. Let A and B be square $n \times n$ matrices such that AB is invertible. Are A and B invertible? Explain. [3 marks]

A2. a. Put $(3-2i)(6+i)$ into the form $a+bi$. [3 marks]

b. Evaluate $\left|\frac{3+4i}{2i}\right|$. [3 marks]

c. Find all roots of $z^3 - 11z^2 + 44z - 60 = 0$, given that 3 is one of the roots. [6 marks]

A3. Let $P = (2,-4,-1)$, $Q = (3,-5,5)$ and $R = (6,-7,-2)$ be three points in $\mathbb{R}^3$.

a. Find the distance between P and Q. [2 marks]

b. Find a vector equation for the line through P and Q. [2 marks]

c. Are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ orthogonal? [2 marks]

d. Give an example of a plane in $\mathbb{R}^3$ that has $\overrightarrow{PQ}$ as a normal vector. [2 marks]

A4. For what values of a is $\{(1,1,0),(1,0,-2),(a,1,-a)\}$ linearly independent? [3 marks]

A5. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

a. Find the image of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ under T. [2 marks]

b. Find the image under T of the square whose corners are $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, and sketch the image. [5 marks]

A6. Let $A = \begin{pmatrix} 1 & -3 \\ -6 & 8 \end{pmatrix}$.

a. Find all eigenvalues for A. [3 marks]

b. Find an eigenvector for each eigenvalue of A. [3 marks]

c. Show that A^2 has the same eigenvectors as A. What are the eigenvalues of A^2? [3 marks]

A7. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 4z = 0\}$. Find a basis for W. [3 marks]

A8. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

a. Find bases for the kernel and range of T. [4 marks]

b. Find the rank and nullity of T. [1 mark]

SECTION B

B1. a. Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 1 \\ -4 & -12 & c \end{vmatrix}.$$

[4 marks]

b. Show that the matrix $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 8 & 1 \\ -4 & -12 & c \end{pmatrix}$ is invertible provided $c \neq -4$, and find the inverse when $c = -2$. [7 marks]

c. Use the inverse matrix found in part (b) to solve the system of equations

(i)

$$\begin{aligned} x + 2y - z &= 3 \\ 3x + 8y + z &= 4 \\ -4x - 12y - 2z &= 0 \end{aligned}$$

(ii)

$$\begin{aligned} x + 2y - z &= 1 \\ 3x + 8y + z &= 1 \\ -4x - 12y - 2z &= 2 \end{aligned}$$

[4 marks]

d. Find the value of d such that the system of equations

$$\begin{aligned} x + 2y - z &= 4 \\ 3x + 8y + z &= 6 \\ -4x - 12y - 4z &= d \end{aligned}$$

has solutions, and for that value of d, determine the general solution. [7 marks]

e. In the case where the system of equations in part (d) has solutions, interpret that system geometrically. [3 marks]

B2. Let $P = (4, 5, 5)$, $Q = (2, -2, 4)$, and $R = (-1, -4, -6)$ be points in $\mathbb{R}^3$.

a. Show that the equation of the plane containing the three points P, Q, and R is $4x - y - z = 6$. [6 marks]

b. Find parametric equations for the line of intersection of the plane found in (a) with the plane $3x - y + 3z = 6$. [6 marks]

c. Find the angle between the vector $\overrightarrow{PQ}$ and the vector $\underline{v} = (2, 1, -2)$. [5 marks]

d. Give an example of a line whose direction is orthogonal to the plane found in (a). [3 marks]

e. Give an example of a plane that has no points in common with the plane found in (a). [3 marks]

f. Give an example of a plane that intersects the plane found in (a). [2 marks]

B3. a. On an island there are two competing mobile phone service providers, Xodafone and Yellow. Let x_i be the number of Xodafone customers in year i, and let y_i be the number of Yellow customers in year i. Let $M = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$. The movement of customers between companies from one year to the next is given by

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

(i) Find the characteristic polynomial of M, and show that M has a steady state vector. [3 marks]

(ii) Find a steady state vector for M. [2 marks]

(iii) Find the other eigenvalue and eigenvector for M. [2 marks]

(iv) Find a diagonal normal form for M. [2 marks]

b. Suppose that in January 2007, Xodafone has 100,000 customers and Yellow has 200,000 customers.

(i) How many customers will each company have in January 2008 and January 2009? [3 marks]

(ii) What *proportion* of the customers that were with Xodafone in January 2007, have switched to Yellow by January 2008? [3 marks]

(iii) *How many* of the customers that were with Xodafone in January 2007 have switched to Yellow by January 2008? [2 marks]

(iv) What happens to the vector $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ as n goes to $+\infty$? [5 marks]

(v) Interpret your answer to (iv) in terms of the number of customers with each company. [3 marks]

B4. a. Let W be the set of points in $\mathbb{R}^3$ parametrised by

$$x = 3t, \quad y = 4t - 4, \quad z = -t, \quad t \in \mathbb{R}.$$

Is W a subspace of $\mathbb{R}^3$? [2 marks]

b. Let $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Find the image of the line $y = x + 1$ under T_A. [5 marks]

c. (i) Show that the quadratic form $5x^2 - 2xy + 5y^2$ can be represented by $\underline{v} \cdot (M\underline{v})$, where $M = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. [2 marks]

(ii) Find the two eigenvectors $\underline{v}_1$ and $\underline{v}_2$ of M with length 1. [6 marks]

(iii) The two eigenvectors from part (ii) form a new basis $\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$ for $\mathbb{R}^2$. Is your new coordinate system a rotation or a reflection of the old (x, y)-coordinate system? [3 marks]

(iv) Determine the normal form of the conic section

$$5x^2 - 2xy + 5y^2 = 36.$$

[4 marks]

(v) Is it an ellipse or hyperbola? At which points does it cut the x' and y' axes? (You do not need to sketch it.) [3 marks]