1. TYPE OF DEGREE: BSc.

2. SESSION: May 2008.

3. MODULE CODE: MA1920.

4. MODULE TITLE: Linear Algebra.

5. TIME ALLOWED: **TWO hours** plus **five minutes** reading time.

6. a. This paper consists of TWO SECTIONS.
   Answer ALL questions of SECTION A.
   Answer TWO questions out of FOUR from SECTION B. If more than TWO questions from Section B are attempted, only the marks from the best TWO solutions will be counted.
   Section A carries 50% of the total marks for the paper. Each question in Section B carries equal marks. An indication of the marks allocated to each sub-section of a question is shown in square brackets in the right hand margin.

7. ADDITIONAL INFORMATION: STANDARD FORMULAE:
   The matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents an anticlockwise rotation of angle θ about the origin. The matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ represents a reflection across the line $y = (\tan\theta)x$. The matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ represents a shear in the x-direction of factor k. The matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ represents a shear in the y-direction of factor k.

SECTION A

A1. Let A and B be square $n \times n$ matrices such that $\det(A) = 5$ and $\det(B) = -2$.

a. Do we have enough information to compute $\det(AB)$? If so, find $\det(AB)$. [2 marks]

b. Do we have enough information to compute $\det(A + B)$? If so, find $\det(A + B)$. [2 marks]

A2. a. Evaluate the modulus $|-1-4i|$. [2 marks]

b. Put the complex number $\frac{3-i}{1-i}$ into the form $a + bi$. [3 marks]

c. Evaluate the product $3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \cdot 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ and express your answer in polar form. [2 marks]

d. Find all complex numbers z such that $z^5 = \sqrt{32}$ and express them in polar form. [6 marks]

A3. Let $P = (3, 0, -2)$ and $Q = (5, 2, -1)$ be points in $\mathbb{R}^3$.

a. Find the distance between P and Q. [2 marks]

b. Give a vector equation for the line L containing P and Q. [2 marks]

c. Give an example of a point that lies on the line L from part (b), that is different from P and Q. [2 marks]

A4. a. For what values of a is $\{(1, 1, a), (1, a, 1), (1, a, a)\}$ a basis for $\mathbb{R}^3$? [3 marks]

b. Let $\{\underline{u}, \underline{v}, \underline{w}, \underline{x}\}$ be a set of vectors that spans $\mathbb{R}^3$ such that

$$2\underline{u} + 0\underline{v} + 0\underline{w} - \underline{x} = \underline{0}.$$

List all subsets of $\{\underline{u}, \underline{v}, \underline{w}, \underline{x}\}$ that are bases for $\mathbb{R}^3$. [3 marks]

A5. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by a shear S in the x-direction of factor -2 followed by a reflection R in the axis $y = x$.

a. Write down the matrices representing the transformations S and R. [2 marks]

b. Find the matrix representing T. [2 marks]

c. The unit square is the square whose corners are the points $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. Sketch the image of the unit square under T. [4 marks]

A6. Let $A = \begin{pmatrix} -2 & 5 \\ 6 & -3 \end{pmatrix}$. Find all eigenvalues and eigenvectors for A. [5 marks]

A7. Let M be an invertible square matrix. Show that if $\underline{v}$ is an eigenvector for M of eigenvalue λ, then $\underline{v}$ is an eigenvector of M^{-1} with eigenvalue $\frac{1}{\lambda}$. [4 marks]

A8. Let $W = \{(x, y, z) \in \mathbb{R}^3 : 3x + y - z = 0\}$. Show that W is a vector subspace of $\mathbb{R}^3$. [4 marks]

SECTION B

B1. a. Determine all possible values of the constants c and d such that the system of equations

$$\begin{aligned} -x - \ 2y + \quad z &= 1 \\ 6x + 11y - 2z &= -1 \\ 2x + \ \ 7y + cz &= d \end{aligned}$$

has:

(i) exactly one solution;

(ii) no solution;

(iii) infinitely many solutions.

In the cases where the system has solutions, determine the full set of solutions. [16 marks]

b. Square matrices A and B are called *similar* if there exists an invertible matrix M such that $B = M^{-1}AM$.

(i) Show that if A and B are similar, then $\det(A) = \det(B)$. [5 marks]

(ii) Let A and B be similar, and let B and C be similar. Show that A and C must also be similar. [4 marks]

B2. a. Let $P = (2, 1, 0)$, $Q = (-4, -9, 1)$ and $R = (0, 4, -6)$ be points in $\mathbb{R}^3$. Let l_1 and l_2 be the lines given by the vector equations

$$l_1: \quad (x, y, z) = (5, 0, 6) + \alpha(1, -1, 2),$$

$$l_2: \quad (x, y, z) = (1, 1, 3) + \beta(-2, -1, 1).$$

(i) Find an equation of the form $ax + by + cz = d$ for the plane that contains the points P, Q and R. [6 marks]

(ii) The lines l_1 and l_2 intersect at a point S. Find the coordinates of S. [4 marks]

(iii) Does S lie on the plane found in part a(i)? [2 marks]

(iv) Find the angle between l_1 and l_2. [4 marks]

(v) Find an example of a line that lies within the plane found in part a(i) and give the vector equation of that line. [2 marks]

b. (i) Let $\underline{u} = (5, 4, 1)$, $\underline{v} = (1, 2, -2)$ and $\underline{w} = (-1, -1, 6)$ be vectors in $\mathbb{R}^3$. Are the elements of $\{\underline{u}, \underline{v}, \underline{w}\}$ linearly independent? Does $\{\underline{u}, \underline{v}, \underline{w}\}$ span $\mathbb{R}^3$? [4 marks]

(ii) How many vectors are there in a basis for $\mathbb{R}^3$? [1 mark]

(iii) Let $\{\underline{a}, \underline{b}, \underline{c}\}$ be a set of vectors in which *no* vector can be written as a linear combination of the others. Are the elements of $\{\underline{a}, \underline{b}, \underline{c}\}$ linearly independent? [2 marks]

B3. a. A country consists of two islands, Algebra Island and Geometry Island. Let x_i be the population of Algebra Island in decade i, and let y_i be the population of Geometry Island in decade i. Let $M = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.9 \end{pmatrix}$. The populations of Algebra Island and Geometry Island from one decade to the next are given by $\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$.

(i) Find the eigenvalues for M. [3 marks]

(ii) Find the eigenvectors for M. [4 marks]

(iii) Find a diagonal normal form for M. [2 marks]

(iv) Using part a(iii), find a formula for M^n. [6 marks]

b. Right now, Algebra Island has a population of 2.2 million and Geometry Island has a population of 1.4 million.

(i) How many people will each island have one decade from now? [2 marks]

(ii) How many people will the whole country have 30 years from now? [2 marks]

(iii) Describe the asymptotic behaviour of the vector $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ as n grows large. [6 marks]

B4. a. (i) Write down the quadratic form $6x^2 - 4xy + 3y^2$ using a real symmetric matrix A. [3 marks]

(ii) Find the normal form of the quadratic equation $6x^2 - 4xy + 3y^2 = 28$ and identify the type of conic. Roughly sketch the conic indicating where it cuts the new x' and y' axes. [12 marks]

b. Let $M = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -5 & 2 \end{pmatrix}$.

(i) Find bases for the kernel and range of the transformation $T_M : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ induced by M. [8 marks]

(ii) Find the rank and nullity of T_M. [2 marks]