SECTION A

1. Let $M = \begin{pmatrix} 1-a & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

   a. Evaluate the determinant of M. [2 marks]

   b. For what values of a is M invertible? [2 marks]

2. a. (i) Evaluate the modulus $|2-3i|$, [2 marks]

   (ii) Put the complex number $\frac{3+4i}{2i}$ into the form $a+bi$. [3 marks]

   b. Find all complex numbers z such that $z^3 = 8i$. [6 marks]

3. Let $P = (-2,-5,0)$, $Q = (3,7,2)$, and $R = (0,3,2)$ be points in $\mathbb{R}^3$.

   a. Find the distance between P and Q, and find the distance between P and R. [3 marks]

   b. Find the dot product of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. [2 marks]

   c. Find the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. [2 marks]

4. Let $\underline{u} = (2,2,-1)$, $\underline{v} = (2,-2,1)$ and $\underline{w} = (5,-2,1)$ be vectors in $\mathbb{R}^3$.

   a. Express $\underline{w}$ as a linear combination of $\underline{u}$ and $\underline{v}$. [3 marks]

   b. Is $\{\underline{u}, \underline{v}, \underline{w}\}$ linearly independent? Explain. [1 mark]

   c. Is $\{\underline{u}, \underline{v}, \underline{u}+\underline{v}+\underline{w}\}$ linearly independent? Explain. [3 marks]

5. Let $A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}$.

   a. Find the two eigenvalues of A. [3 marks]

   b. Find an eigenvector of A for each of the eigenvalues found in (a). [3 marks]

6. The linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is given by:

   - a shear S in the x-direction of factor 4, followed by
   - an anticlockwise rotation R of angle $90°$, followed by
   - an expansion E in the y-direction of factor 2.

   a. Write down the matrices representing the transformations S, R, and E. [3 marks]

   b. Find the matrix representing T. [2 marks]

   c. The unit square is the square whose corners are the points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. Find the image of the unit square under T. [3 marks]

7. Let W be the set of points (x, y, z) in $\mathbb{R}^3$ parameterised by

$$x = 4t, \quad y = t + 2, \quad z = -t^2.$$

   Is W a subspace of $\mathbb{R}^3$? [2 marks]

8. Let $T_A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the linear transformation induced by $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$.

   a. Find bases for the kernel and range of T_A. [4 marks]

   b. Find the rank and nullity of T_A. [1 mark]

SECTION B

1. a. Let A be an $n \times n$ matrix and $\underset{\sim}{b} \in \mathbb{R}^n$. If $A\underset{\sim}{x} = \underset{\sim}{b}$ has no solution, what do we know about A? [3 marks]

   b. Determine all possible values of the constants c and d such that the system of equations

$$\begin{aligned} 2x + 4y + z &= -5 \\ x - y - 2z &= -3 \\ 4x + 2y + cz &= d \end{aligned}$$

   has:

   (i) exactly one solution;

   (ii) no solution;

   (iii) infinitely many solutions.

   In the cases where the system has solutions, determine the full set of solutions. [16 marks]

   c. Interpret the three cases (bi), (bii) and (biii) geometrically. [6 marks]

2. Let $P = (1, -2, 4)$, $Q = (0, 3, 1)$ and $R = (-2, 0, 2)$. Let l be the line given by the vector equation $(-2, 1, 1) + t(1, 2, -1)$.

   a. Find an equation for the line through the points P and Q. [3 marks]

   b. Find an equation for the plane through the points P, Q and R. [6 marks]

   c. Find the point of intersection of the line l with the plane found in part (b). [3 marks]

   d. Find the angle between the normal to the plane from (b) and the line l. [4 marks]

   e. Find the vector cross product of $(1, 2, -1)$ with the normal to the plane from part (b). [3 marks]

   f. Show that the vector found in (e) is parallel to the plane from (b). [3 marks]

   g. Give an example of a plane that has no points in common with the plane from (b). [3 marks]

3. a. Write down the following quadratic form using a real symmetric matrix A (d is a *non-zero* constant).

$$x^2 + 4xy + y^2 = d$$

[3 marks]

b. Find the normal form of the quadratic equation and identify the type of conic. How does the type of conic change for positive and negative values of $d \neq 0$? [10 marks]

c. Briefly describe the change of coordinates you used in (b). If it was a rotation, what was the angle? If it was a reflection, what was the axis of reflection? [5 marks]

d. Roughly sketch the conics for $d > 0$ and for $d < 0$, indicating where they cut the new x' and y' axes. [4 marks]

e. Briefly explain one way that you might have chosen a different change of coordinates to the one you chose in (c). [3 marks]

4. Let $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -1 & 4 \\ 7 & 1 & 2 \end{pmatrix}$.

a. Find the kernel and range for the transformation $T_A : \mathbb{R}^3 \mapsto \mathbb{R}^3$ induced by A. [6 marks]

b. Find bases for the kernel and range of T_A. [4 marks]

c. Find the image of the plane $3x + 4y - 7z = 0$ under T_A. [5 marks]

d. If $T_1 : \mathbb{R}^2 \mapsto \mathbb{R}^4$ has rank 2, what is it's nullity? [2 marks]

e. If $T_2 : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is not invertible, what do we know about the dimension of its kernel? [2 marks]

f. Suppose that $\{\underline{u}, \underline{v}, \underline{w}\}$ is a linearly independent set of vectors. For what real values of a is $\{a\underline{u} - \underline{v}, a\underline{v} - \underline{w}, a\underline{w} - u\}$ linearly independent? [4 marks]

g. Suppose that $\{\underline{u}, \underline{v}, \underline{w}, \underline{x}\}$ is a linearly independent set of vectors in $\mathbb{R}^5$. Do we know if $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent? Explain briefly. [2 marks]