1. TYPE OF DEGREE: BSc.
2. SESSION: May 2009.
3. MODULE CODE: MA1915.
4. MODULE TITLE: Calculus and Numerical Methods.
5. TIME ALLOWED: 3 hours (plus 5 minutes reading time).
6. a. NUMBER OF QUESTIONS: A1–10, B1–4, Full marks: All of A and 2 of B.

   b. (LEVEL 1 MOSTLY): Section A / B
7. ADDITIONAL INFORMATION: None

Section A

A1. Without using L'Hôpital's rule, evaluate:

a. $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1}$; [3 marks]

b. $\lim_{x \to \infty} \dfrac{x^3 + 3x^2 + 2}{6x^3 + 3x - 1}$. [2 marks]

A2. a. Let $f(x) = \cos(1/x)$. Find $f'(x)$. [3 marks]

b. Let $g(x) = \sin(3x)$. Find $g'''(x)$. [3 marks]

A3. Solve the inequality $|3x - 1| > 2$. Express your answer using interval notation.

[3 marks]

A4. Sketch the curve given by the following equation:

$$x^2 + y^2 + 3x - 4y = -\frac{9}{4}.$$

[3 marks]

A5. Find the natural domain of the function rule:

$$f(x) = \frac{1}{\sqrt{x-2} - 3}.$$

[3 marks]

A6. a. Express $|x - 3|$ using a piecewise definition. [2 marks]

b. Hence evaluate $\displaystyle\int_0^4 |x - 3|\,\mathrm{d}x$. [3 marks]

A7. Determine whether the following series are convergent or divergent:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{3/2}}{\sqrt{n^3 + 3}}$;

b. $\displaystyle\sum_{n=1}^{\infty} \frac{\tanh(n)}{n!}$.

[6 marks]

A8. Given that

$$F(x, y) = xy(x^2 + y^2),$$

where $x = r\cos\theta$ and $y = r\sin\theta$, use the **Chain rule** to determine $\dfrac{\partial F}{\partial r}$ and $\dfrac{\partial F}{\partial \theta}$. Express your answer in terms of r and θ. [7 marks]

A9. Using Lagrange polynomials, fit an interpolating polynomial through the points $f(-0.2) = 1.2$, $f(0.2) = 0.7$, $f(0.5) = 1.7$, and hence give an estimate of $f(0.0)$. [6 marks]

A10. Apply the Composite Simpsons Rule (CSR) with N=2 to approximate the integral

$$\int_1^2 \cos(x^2)\, dx$$

and estimate the error. Hint: $|\int_a^b f\, dx - S_N f| \leq \frac{(b-a)^5}{2880N^4} \max_{x\in[a,b]} |f^{(4)}(x)|$. [6 marks]

Section B

B1. a. Determine the exact value of the following integrals:

(i) $\displaystyle\int_0^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x$; [4 marks]

(ii) $\displaystyle\int_0^{1} x\sqrt{3+x^2}\,\mathrm{d}x$. [4 marks]

b. Find $\displaystyle\int \frac{x^2}{x^2-4x+3}\,\mathrm{d}x$. [4 marks]

c. For $n \geq 0$, let I_n be given by

$$I_n = \int_0^2 x^n e^{-x}\,\mathrm{d}x.$$

(i) Determine the exact value of I_1. [3 marks]

(ii) By integrating by parts, show that for $n \geq 1$,

$$I_n = -\frac{2^n}{e^2} + nI_{n-1}.$$

[3 marks]

(iii) Hence find the exact value of I_3. [2 marks]

d. Find $\displaystyle\int_0^{\infty} xe^{-x}\,\mathrm{d}x$.

[You may assume that $\lim_{x\to\infty} xe^{-x} = 0$.] [5 marks]

B2. a. Given that $3x + 2y^3 = xy$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $x = -1$, $y = 1$. [8 marks]

b. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin^{-1}(\cos x)$.

(i) Evaluate $g(\pi)$ and $g(2\pi)$. [2 marks]

(ii) Find $g'(x)$.
[Hint: $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$.] [2 marks]

(iii) What is the range of g'? [1 mark]

(iv) Solve the inequality $g'(x) > 0$. [2 marks]

c. Determine $\lim_{x \to \infty}(x^2 - 2x + 2)e^{-x}$. [5 marks]

d. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 9x^2 + 24x + 7$. Find the intervals of increase and decrease of f and its local minima and maxima. [5 marks]

B3. a. (i) Show that the Taylor series expansion for the function $f(x, y) = (4 + xy)^{3/2}$ about the point $(0, 1)$, is given by

$$f(x, y) = 8 + 3xy + \frac{3x^2}{16} + \ldots .$$

NB. Take the positive square root where appropriate. [8 marks]

(ii) Use part (i) to evaluate approximately the repeated integral

$$\int_0^1 \int_0^1 (4 + xy)^{3/2} \, \mathrm{d}x \, \mathrm{d}y.$$

[5 marks]

b. Find the general solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9y = 3x + \cos(2x).$$

[12 marks]

B4. Consider the initial value problem (IVP)

$$y' = t^2 e^{-t} - 2y, \quad t \geq a$$
$$y(a) = y_0.$$

a. Write a Matlab function `ImpEuler` which solves approximately the above IVP on the interval $[a, b]$ using the improved Euler's method. The inputs should be a, b, y_0 and N, the number of steps. The output should be y_N, where y_N is the estimate of $y(b)$ obtained by the improved Euler's method with step size $h = (b-a)/N$. (Do not bother to include comment statements). [10 marks]

b. Let $a = 0$ and $y_0 = 0$ then using the improved Euler's method, first with $h = 1$ and second with $h = 0.5$, calculate two estimates for $y(1)$. [10 marks]

c. Given that the exact solution of the IVP in (b) is

$$y(t) = (t^2 - 2t + 2)e^{-t} - 2e^{-2t}$$

make a statement regarding the difference in error when using $h = 1$ and $h = 0.5$. [3 marks]

d. What order of the error do we expect to see when using the improved Euler's method? [2 marks]