1. TYPE OF DEGREE: BSc.

2. SESSION: May 2006.

3. MODULE CODE: MA1915.

4. MODULE TITLE: Calculus and Numerical Methods.

5. TIME ALLOWED: **THREE hours** plus **five minutes** reading time.

6. a. NUMBER OF QUESTIONS: Part A has 7 questions, Part B has 4 questions. Full marks: 100.

   b. Answer **ALL** questions from **SECTION A**.
   Answer **TWO** questions from **SECTION B**. If more than **TWO** questions from **SECTION B** are answered, marks from the best **TWO** answers will be counted.

   Section A carries 50% of the total marks available for the paper.

   All questions in **Section B** carry equal marks.

   An indication of the marks allocated to each sub-section of a question is shown in brackets in the right hand margin.

7. ADDITIONAL INFORMATION: Calculators: Casio fx 82, Casio fx 83 and Casio fx 85 ONLY.

A1. a. Determine whether the series below is convergent or divergent. If convergent, find the sum

$$\sum_{n=0}^{\infty} \frac{2^{n+3}}{3^{n+1}} \; .$$

[3 marks]

b. Evaluate the sum

$$\sum_{n=2}^{100} \frac{1}{(n+1)(n+2)} \; .$$

[3 marks]

A2. Evaluate the limits

a.

$$\lim_{x \to \infty} \frac{3x^2+5}{2x^2-6} \; ,$$

[3 marks]

b.

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin^2 x} \; .$$

[4 marks]

A3. a. Show that

$$\frac{d}{dz}(\tan z) = 1 + \tan^2 z \; .$$

b. Use the result of (a.) and implicit differentiation to determine dy/dx, when $y = \tan^{-1} x$.

c. Hence, or otherwise, evaluate

$$\lim_{x \to 0} \frac{\tan^{-1} x}{x} \; .$$

[5 marks]

A4. a. Evaluate the following definite integrals:

(i)

$$\int_{0}^{\sqrt{\pi}} x\sin(x^2)\,dx\,,$$

[3 marks]

(ii)

$$\int_{0}^{1/2} xe^{2x}\,dx\,.$$

[3 marks]

b. Present the function $|x-2|$ in piecewise form and evaluate the following definite integral

$$\int_{1}^{4} |x-2|\,dx\,.$$

[5 marks]

A5. Show that the function

$$f(x,y) = x^3 - 12xy + 8y^3$$

has a critical point at $(0,0)$. Determine the second critical point and classify both. [8 marks]

A6. Explain why the equation

$$\cos x - x^2 = 0$$

must have a solution in the interval $[0,1]$. [2 marks]

Find this solution, correct to ± 0.2, using the bisection method. [5 marks]

A7. By formulating the Lagrange polynomials, fit an interpolating polynomial through the points $f(0.2) = 1.76$ and $f(0.3) = 1.58$ and hence give an estimate of $f(0.4)$. [6 marks]

B1. Consider the functions

$$f(x) := x^3 - 3x^2 - x + 3\,; \qquad\qquad g(x) := \frac{1}{f(x)}\,.$$

a. Factorise $f(x)$ and hence determine all the roots of the equation $f(x) = 0$. [5 marks]

b. What are the natural domain and codomain of f and g? [4 marks]

c. Determine the coordinates of the critical points of f and classify them.
Explain your answers. [4 marks]

e. Determine the root of $f''(x) = 0$.
What is the value of f at this point? Is this a point of inflection?
Explain your answer. [2 marks]

f. Sketch the graphs of f and g on the same axes.
Show and explain all relevant features of the curves. [7 marks]

g. Evaluate

$$\int_{-1}^{3} f(x)\,dx\,.$$

What is the geometric meaning of this result? [3 marks]

B2. a. Solve the differential equation

$$\frac{dy}{dx} + y = x\,, \qquad \text{where} \qquad y(0) = 1\,.$$

[6 marks]

b. Use the substitution $y = vx$ to solve the homogeneous differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}\,, \qquad \text{where} \qquad y(1) = 1\,.$$

Express your answer in the form $y = f(x)$. [8 marks]

c. Find the solution of the initial-value problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x\,, \qquad \text{where} \qquad y(0) = -\frac{4}{9}, \qquad y(1) = \frac{8}{9}\,.$$

[11 marks]

B3. The Taylor series for a function of two variables, $f(x,y)$, about the point (a,b) is given by

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{(x-a)^2}{2}f_{xx}(a,b) + (x-a)(y-b)f_{xy}(a,b) + \frac{(y-b)^2}{2}f_{yy}(a,b) + \ldots$$

a. Obtain the Taylor series expansion, up to and including all second derivatives, for the function $f(x,y)$ about the point $(0,1)$, when

$$f(x,y) = ye^{2xy}.$$

[11 marks]

b. Use the method of Lagrange multiplier to locate the critical points of the function $f(x,y) = 3x+4y$ subject to the constraint $x^2+y^2 = 1$. Determine the nature of these critical points. [14 marks]

B4. Consider the initial value problem (IVP)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 - 2y, \qquad x \geq x0 \\ y(x0) &= y0. \end{aligned}$$

a. Write a matlab function **Eulers method** which solves approximately the above IVP on the interval $[x0, b]$. The inputs should be $x0, y0, b$ and h, the step size, you may assume that h divides $b - x0$ exactly. The outputs should be $\mathbf{x} = (x0, x1 \ldots xn)$ and $\mathbf{y} = (y0, y1 \ldots yn)$, where $xn = x0 + nh$ and yn is the estimate of $y(xn)$ obtained by Euler's method with step length h. (Do not bother to include comment statements). [10 marks]

b. Let $x0 = 0$ and $y0 = 0$ then using Euler's method, first with $h = 0.1$ and second with $h = 0.05$, calculate two estimates for $y(0.3)$. [8 marks]

c. State whether Euler's method is an explicit or an implicit formula, giving an explanation for your answer. [2 marks]

d. (i) Given that the exact solution of the IVP in (b) is

$$y(x) = \frac{x^2}{2} - \frac{x}{2} - \frac{e^{-2x}}{4} + \frac{1}{4},$$

make a statement regarding the difference in error when using $h = 0.1$ and $h = 0.05$. [3 marks]

(ii) What order error do we expect to see when using Euler's method? [2 marks]