1. TYPE OF DEGREE: BSc.
2. SESSION: May 2008.
3. MODULE CODE: MA1915.
4. MODULE TITLE: Calculus and Numerical Methods.
5. TIME ALLOWED: 3 hours (plus 5 minutes reading time).
6. a. NUMBER OF QUESTIONS: Part A has 7 questions, Part B has 4 questions. Full Marks: 100.

   b. Answer **all** questions from **Section A**. Answer **two** questions from **Section B**. If more than **two** questions from **Section B** are answered, marks from the best **two** answers will be counted. Then **Section A** carries 50 % of the total marks available for the paper, and all questions in **Section B** carry equal marks.

   An indication of the marks allocated to each sub-section of a question is shown in brackets in the righthand margin.
7. ADDITIONAL INFORMATION: Calculators: Casio fx 82, Casio fx 83 and Casio fx 85 ONLY.

A1. Evaluate the limits

a.

$$\lim_{x\to 1} \frac{x^2+3x-4}{2x^2-x-1},$$

[3 marks]

b.

$$\lim_{x\to 0} \frac{\cos(2x)-1}{\cos(3x)-1}.$$

[4 marks]

A2. a. Differentiate

(i) $$f(x) = x^3 \ln(2x),$$ [2 marks]

(ii) $$g(x) = \cos(3x^4 - 2x).$$ [2 marks]

b. (i) Determine the natural domain of the function

$$h(x) = \frac{1}{\sqrt{1+\sin x}}.$$

[2 marks]

(ii) Differentiate the function

$$h(x) = \frac{1}{\sqrt{1+\sin x}}.$$

[2 marks]

A3. Evaluate the following definite integrals:

a.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\, dx,$$

[3 marks]

b.

$$\int_0^1 (x+3)e^{2x}\, dx,$$

[3 marks]

c.

$$\int_{\frac{1}{e}}^{e} |\ln x|\, dx.$$

[3 marks]

A4. Use an appropriate test to determine whether the following series converge or diverge:

a.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \,,$$

[3 marks]

b.

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{1+e^{-n}} \,.$$

[3 marks]

A5. Determine the first three non-zero terms in the MacLaurin series of $f(x) = e^{-x^2}$.

Hence, evaluate approximately the integral

$$\int_0^{1/2} e^{-x^2}\, dx$$

and obtain the value of

$$\lim_{x \to 0} \frac{e^{-x^2} - 1 + x^2}{x^4}.$$

[6 marks]

A6. a. Explain why the equation

$$\ln\left(\frac{x+1}{2}\right) + x^2 = 0$$

must have a solution in the interval $[0, 1]$. [2 marks]

b. Find this solution, correct to ± 0.1 using Newton's method, with $x_0 = 1$.

[5 marks]

A7. Using Lagrange polynomials fit an interpolating polynomial through the points $f(0.2) = 1.4$, $f(0.4) = 0.9$, $f(0.6) = 1.7$ and hence give an estimate of $f(0.5)$. [7 marks]

B1. Consider the function

$$f\ :[0,2\pi]\to\mathbb{R},\ f(x):=\sin^2 x-3\sin x+2.$$

a. Determine the roots of the equation $f(x)=0$. Compute $f(0)$, $f(\pi)$ and $f(2\pi)$. [4 marks]

b. Determine the coordinates of the critical points of f and classify them. Explain your answers. What are the absolute minima and maxima of f ? [5 marks]

c. Compute the roots of the equation $f''(x)=0$ and the values of f at these points. Are these points of inflection? Explain your answers. [5 marks]

d. Sketch the graph of f indicating the above features in your graph. [4 marks]

e. Give the complete function specification (domain, codomain, rule) for the composition $g\circ f$ when

$$g:\ (0,\infty)\to\mathbb{R},\quad g(x):=\ln x$$

and f is given above. Determine the coordinates of the absolute maxima and minima of $g\circ f$ if they exist. Does $g\circ f$ have any asymptotes? If yes, give their equation(s). Sketch the graph of $g\circ f$ indicating the above features in your graph. [7 marks]

B2. a. Use an integrating factor to obtain the solution to:

$$\frac{dy}{dx}-2xy=e^{x^2},\qquad y(0)=1.$$

[6 marks]

b. Find the general solution to:

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=2\cos(2x)\ .$$

[12 marks]

c. A curve in $\mathbb{R}^2$ is mapped by the position vector

$$\mathbf{r}=t\cos^2 t\mathbf{i}+t\sin 2t\mathbf{j},\quad 0\le t\le 2\pi.$$

Use parametric differentiation to determine the gradient of the curve at the point $t=\pi/4$. Hence, determine the Cartesian equation of the tangent line to the curve at the point $t=\pi/4$. [7 marks]

B3. A surface is specified by $z = f(x, y)$ where $f(x, y) = x^3 + 3xy^2 - 6xy$.

a. The surface has four critical points, two of which are located at $(0, 0)$ and $(1, 1)$. Classify these critical points and locate the other two. [13 marks]

b. Find the normal vector and tangent plane to the surface at the point $(-1, 0, -1)$. [6 marks]

c. Determine the volume enclosed by the above tangent plane, the xy-plane and the four planes $x = 0$, $x = 2$, $y = 0$, $y = 1$. [6 marks]

B4. Consider the initial value problem (IVP)

$$\begin{aligned} y' &= te^{-3t} - y, \quad t \geq t_0 \\ y(t_0) &= y_0. \end{aligned}$$

a. Write a Matlab function `Euler` which solves approximately the above IVP on the interval $[t_0, b]$ using Euler's method. The inputs should be t_0, y_0, b and n, the number of steps. The outputs should be $t = (t_0, t_1, \ldots, t_n)$ and $y = (y_0, y_1, \ldots, y_n)$, where $t_i = t_0 + i \cdot h$, $h = (b - t_0)/n$ and y_i is the estimate of $y(t_i)$ obtained by Euler's method with step size h. (You do not need to include comment statements). [10 marks]

b. Let $t_0 = 0$ and $y_0 = 0$ then using Euler's method, first with $h = 0.1$ and second with $h = 0.05$, calculate two estimates for $y(0.2)$. [10 marks]

c. Given that the exact solution of the IVP in (b) is

$$y(t) = -\frac{1}{2}te^{-3t} - \frac{1}{4}e^{-3t} + \frac{1}{4}e^{-t}.$$

Compute the errors when using $h = 0.1$ and $h = 0.05$. Comment on your results. [3 marks]

d. What order of error do we expect for Euler's method? [2 marks]