1. TYPE OF DEGREE: BSc.

2. SESSION: May 2007.

3. MODULE CODE: MA1915.

4. MODULE TITLE: Calculus and Numerical Methods.

5. TIME ALLOWED: 3 hours (plus 5 minutes reading time).

6. a. NUMBER OF QUESTIONS: Part A has 7 questions, Part B has 4 questions. Full Marks: 100.

   b. Answer **all** questions from **Section A**. Answer **two** questions from **Section B**. If more than **two** questions from **Section B** are answered, marks from the best **two** answers will be counted.

   Section A carries 50 % of the total marks available for the paper.

   All questions in **Section B** carry equal marks.

   An indication of the marks allocated to each sub-section of a question is shown in brackets in the righthand margin.

7. ADDITIONAL INFORMATION: Calculators: Casio fx 82, Casio fx 83 and Casio fx 85 ONLY.

A1. Evaluate the limits

a.

$$\lim_{x \to 0} \frac{1 - e^x}{\sin(2x)},$$

[3 marks]

b.

$$\lim_{x \to \infty} \frac{3x^3 - 1}{5x - 2x^3}.$$

[3 marks]

A2. a. Differentiate

(i) $$f(x) = x^3 e^{2x},$$ [2 marks]

(ii) $$g(x) = \sin(x^5).$$ [2 marks]

b. (i) Differentiate the function

$$h(x) = \ln\left(\frac{2+x}{2-x}\right).$$

[2 marks]

(ii) Determine the natural domain of h. [2 marks]

A3. a. Evaluate the following definite integrals:

(i)

$$\int_0^{\pi/2} \sin x \cos^3 x \, dx,$$

[3 marks]

(ii)

$$\int_0^1 (x-2)e^{-3x} \, dx.$$

[3 marks]

b. Evaluate the following indefinite integral:

$$\int \frac{\sqrt{x+1} - x}{(x+1)^2} \, dx.$$

[3 marks]

A4. Solve the differential equation

$$\frac{dy}{dx} + 2y = 3x, \quad \text{where} \quad y(1) = 0.$$

[6 marks]

A5. Determine the critical points of the function

$$f(x, y) = x^3 - 3xy + 8y^3 + 7$$

and classify them. [8 marks]

A6. Given the initial value problem

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^3 - 2xy^2, \qquad x \geq 1, \\ y(1) &= 2, \end{aligned}$$

carry out **one** step of the Euler predictor-corrector method to get an estimate for $y(1.1)$. [6 marks]

A7. Using Lagrange polynomials fit an interpolating polynomial through the points $f(1.2) = 1.76$, $f(1.4) = 1.52$ and $f(1.6) = 1.38$ and hence give an estimate of $f(1.5)$. [7 marks]

B1. Consider the function
$$f(x) := \frac{x}{x^2 - 5x + 4}.$$

a. Factorise the function $g(x) := x^2 - 5x + 4$ and determine the natural domain of f. Determine the roots of the equation $f(x) = 0$. [3 marks]

b. Compute
$$\lim_{x\to\infty} f(x) \qquad \text{and} \qquad \lim_{x\to-\infty} f(x).$$
[2 marks]

c. Does the graph of f have any vertical or horizontal asymptotes? If yes, what are they? [3 marks]

d. Determine the coordinates of the critical points of f and classify them. Explain your answers. Does f have an absolute maximum or minimum? [5 marks]

e. Show that $f''(x) = 0$ has a root in the interval $(-5, -4)$. Explain your answer. [3 marks]

f. Sketch the graph of f indicating the above features in your graph. [5 marks]

g. Evaluate
$$\int_5^6 f(x)\,dx$$
simplifying your answer as much as possible.

[4 marks]

B2. a. Use the substitution $y = vx$ to solve the homogeneous differential equation
$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}\,, \qquad \text{where} \qquad y(1) = 1\,.$$
Express your answer in the form $y = f(x)$. [9 marks]

b. Find the solution of the initial-value problem
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = x\,, \qquad \text{where} \qquad y(0) = 1, \qquad y(1) = 2\,.$$
[12 marks]

c. Determine whether the series
$$\sum_{n=0}^{\infty} \frac{3^{2n+3}}{4^{2n+1}}$$
is convergent or divergent. If convergent, find the sum. If divergent, explain why. [4 marks]

B3. The Taylor series for a function of two variables, $f(x,y)$, about the point (a,b) is given by

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{(x-a)^2}{2}f_{xx}(a,b) + (x-a)(y-b)f_{xy}(a,b) + \frac{(y-b)^2}{2}f_{yy}(a,b) + \dots$$

a. Obtain the Taylor series expansion, up to and including all second derivatives, for the function $f(x,y)$ about the point $(0,1)$, when

$$f(x,y) = xe^{3xy}.$$

[10 marks]

b. Find a normal vector to the surface $z - xe^{3xy} = 0$ at the point $(0,1,0)$. [3 marks]

c. Use the method of Lagrange multiplier to locate the critical points of the function $f(x,y) = 6x + 7y$ subject to the constraint $x^2 + 2y^2 = 2$. Determine the nature of these critical points. [12 marks]

B4. a. (i) Without sketching a graph explain why the equation

$$0 = 2\cos x - e^x$$

must have a solution on the interval $[0,1]$. [2 marks]

(ii) Now sketch the graph and hence establish the number of roots on the interval $[0,1]$. [6 marks]

b. (i) Write down the formula for Newton's method. [2 marks]

(ii) Using a starting point of $x_0 = 0.5$ and using working calculations of 8 decimal places, estimate the solution, X, of $f(x) = 0$ for $x \in [0,1]$ correct to 0.000001 by Newton's method. [10 marks]

c. (i) Write down the general formula for the composite Simpson's rule. [2 marks]

(ii) Now given that

x	0	0.1	0.2	0.3	0.4	0.5
f(x)	1	0.8848	0.7387	0.5608	0.3503	0.1064

x	0.6	0.7	0.8	0.9	1.0
f(x)	-0.1714	-0.4841	-0.8321	-1.2164	-1.6377

estimate

$$\int_0^1 f(x)\,\mathrm{d}x,$$

using the composite Simpson's rule. [3 marks]