UNIVERSITY OF ABERDEEN **SESSION 2005–06**

Degree Examination
MX4544 Representation Theory of Finite Groups
Thursday 25 May 2006 (3 pm—5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer 3 questions. All questions carry equal weight.

1. Let G be a finite group.

(a) Explain what is meant by a complex-valued class-function of G, and give the definition of the standard inner product on the space of all such class functions.

(b) State the orthogonality relations for complex irreducible characters of G.

(c) Prove that the degree of each complex irreducible character of G divides the order of G.

(You may assume without proof any property of algebraic integers you require, as long as you state clearly the results you are making use of. You may also assume that $[G : C_G(x)]\frac{\chi(x)}{\chi(1)}$ is an algebraic integer for each $x \in G$.)

2. **(a)** Let H be a non-Abelian finite group containing a non-identity element x such that $[H : C_H(x)]$ is a power of a prime. Prove that H is not simple.

(b) Let G be a finite group of order $p^a q^b$ where p, q are primes and a, b are non-negative integers. Prove that if G is simple, then G has order p or q.

(For parts **(a)** and **(b)**, you may assume without proof any necessary orthogonality relation for group characters you require, and any property of algebraic integers you require, as long as you state clearly the results you are making use of. You may also assume that $[G : C_G(x)]\frac{\chi(x)}{\chi(1)}$ is an algebraic integer for each $x \in G$.)

3. Let G be a finite group.

(a) Describe what is meant by the *ring of generalized characters* of G, explaining how its ring structure arises. (You may quote without proof results from the course which you need to justify your statements.)

(b) Let H be a subgroup of G and let θ be a complex-valued class function of H. Explain how the values of the induced class function $Ind_H^G(\theta)$ are defined.

(c) Prove that if θ is as above, and ψ is any complex-valued class function of G, then

$$\langle Ind_H^G(\theta), \psi \rangle = \langle \theta, Res_H^G(\psi) \rangle.$$

4. **(a)** The finite group G is known to have a subgroup $H \neq G$ such that $H \cap g^{-1}Hg = 1_G$ for all $g \in G \backslash H$.

(i) State Frobenius' Theorem regarding the existence of a certain normal subgroup of G.

(ii) Prove that if θ is a class function of H and h is a non-identity element of H, then the induced class function $Ind_H^G(\theta)$ takes the same value at h as θ does.

(b) Let P be a Sylow p-subgroup of a finite group X for some prime p. Suppose that $|P| = p$ and $N_X(P) = P$. Using the result of part (i) of **(a)**, or otherwise, prove that X has a normal subgroup Y of index p.