UNIVERSITY OF ABERDEEN **SESSION 2006–07**

DEGREE EXAMINATION

MX4544 Representation Theory of Finite Groups

Thursday 24 May 2007 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Let G be a finite group and let

$$\sigma : G \to GL_n(V) \quad \text{and} \quad \tau : G \to GL_m(W)$$

be finite-dimensional representations of G over the field of complex numbers.

(i) Define what it means for σ and τ to be equivalent.

(ii) Define what it means for σ to be irreducible.

(b) Let G be a finite group and let $\chi : G \to \mathbb{C}$ be a complex valued function. In each of the following give a brief reason why χ is not a character of G. You may use results proved in the course on character values as long as you clearly state which result you are using.

(i) $G = S_4$, $\chi((12)) = 3$ and $\chi((13)) = 0$.

(ii) G has an element g such that $g^2 = 1_G$ and $\chi(g) = i$, where i is a complex square root of -1.

(iii) G has an element g such that $\chi(g) = \frac{1}{2}$.

(c) Let G be a finite non-abelian simple group. Prove that the trivial representation is the only one-dimensional complex representation of G. You may use any result on group homomorphisms you wish for this. Also, recall that a finite group G is simple if $\{1_G\}$ and G are the only normal subgroup of G.

2. **(a)** Let G be a finite abelian group. Show that any irreducible complex representation of G is 1-dimensional. You may use any form of Schur's lemma for this and/or the orthogonality relations.

(b) Let

$$D_8 = \langle r, s \mid r^4 = 1,\ s^2 = 1,\ srs^{-1} = r^{-1} \rangle$$

be the dihedral group of order 8. The conjugacy classes of D_8 are:

$$\{1\}, \{r^2\}, \{r, r^3\}, \{s, r^2 s\}, \{rs, r^3 s\}.$$

(i) Show that there is a representation $\tau : D_8 \to GL_2(\mathbb{C})$ satisfying

$$\tau(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) Show that the representation τ is irreducible. State clearly any results you use.

(iii) Prove that any 2-dimensional irreducible representation of D_8 is equivalent to τ. State clearly any results you use.

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(c) Let G be a finite group. Let ρ be the regular representation of G and let χ_ρ be the character of ρ.

(i) Let $g \in G$. What is the value of $\chi_\rho(g)$? You need not give any explanation for your answer.

(ii) Let $\chi_1, \chi_2, \cdots, \chi_k$ be the distinct irreducible characters of G. Write χ_ρ as a linear combination of the χ_i's. You need not give any explanation for your answer.

3. Let G be a finite group.

(a) Give the definition of the standard inner product on the space of complex valued class functions of G and state the orthogonality relations for the irreducible characters of G.

(b) Let $g \in G$ and let $\mathcal{C}$ be the conjugacy class of g. Let $\chi_1, \chi_2, \cdots, \chi_k$ be the distinct irreducible characters of G.

(i) Prove that the characteristic function $1_\mathcal{C}$ of $\mathcal{C}$ satisfies:

$$1_\mathcal{C} = \frac{1}{|C_G(g)|} \sum_{1 \leq i \leq k} \overline{\chi_i(g)} \chi_i,$$

where $C_G(g)$ is the centraliser of g in G.

(ii) Deduce that if every irreducible character of G is real valued, then g and g^{-1} are conjugate for all $g \in G$. You may use the fact that $C_G(g) = C_G(g^{-1})$ for all $g \in G$.

(c) Suppose that G has four conjugacy classes and $x_1 = 1, x_2, x_3, x_4$ is a set of representatives of the conjugacy classes of G. Suppose that G has the following character table:

	$x_1 = 1$	x_2	x_3	x_4
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	A	$\frac{-1-\sqrt{5}}{2}$	C
χ_4	2	$\frac{-1-\sqrt{5}}{2}$	B	D

Fill in the missing entries A, B, C and D. Show your work.

4. **(a)** State Maschke's theorem.

(b) Let G be a finite group and let θ be a character of G such that $\langle \theta, \theta \rangle = 3$. Prove that θ is a sum of three distinct irreducible characters of G. State clearly the results you are using.

(c) Let G be a finite group, let $x_1, x_2, \cdots, x_k$ be the representatives of the distinct conjugacy classes of G and let χ be an irreducible character of G.

(i) Prove that

$$\sum_{1 \leq i \leq k} \frac{|G|}{|C_G(x_i)|} |\chi(x_i)|^2 = |G|.$$

You may use the orthogonality relations for this.

(ii) Use (i) to deduce that $\chi(1)$ is a divisor of $|G|$. You may use the fact that $\frac{|G||\chi(x)|^2}{|C_G(x)|\chi(1)}$ is an algebraic integer for any element $x \in G$. You may use any result on algebraic integers you require as long as the results you use are clearly stated

(d) State Burnsides $p^a q^b$ theorem.