UNIVERSITY OF ABERDEEN **SESSION 2005–06**

DEGREE EXAMINATION

MX4543 Introduction to Lie Algebras

Monday 22 May 2006 (3 pm—5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer 3 questions. All questions carry equal weight.

Please note that full explanations form an essential part of your solution.

1. We consider the following subspace of $sl_3(\mathbb{R})$:

$$L := \left\{ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}.$$

The following elements form a basis of L:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Show that L is a Lie subalgebra of $sl_3(\mathbb{R})$. Determine $[e_i, e_j]$ for $1 \leq i, j \leq 3$.

(b) Show that L is non-abelian and simple (that is, the only ideals are $\{0\}$ and L).

(c) Consider the adjoint representation $\mathrm{ad}\colon L \to gl(L)$. (Recall that $\mathrm{ad}(x)$ is the linear map of L into itself which sends y to $[x, y]$ for any $y \in L$.) Let $x = ae_1 + be_2 + ce_3 \in L$ and determine the matrix of $\mathrm{ad}(x)$ with respect to the basis $\{e_1, e_2, e_3\}$. Show that, if $x \neq 0$ then the characteristic polynomial of that matrix only has one real root, namely 0.

(d) Show that L is not isomorphic to $sl_2(\mathbb{R})$.

(*Hint.* If L were isomorphic to $sl_2(\mathbb{R})$, then there should be $0 \neq e, h \in L$ such that $[h, e] = 2e$, that is, 2 an eigenvalue for $\mathrm{ad}(h)$.)

2. Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Let $\kappa\colon L \times L \to \mathbb{C}$ be the Killing form of L. Recall that $\kappa(x,y) = \mathrm{trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$ for all $x, y \in L$.

(a) Define the notion of a *nilpotent* Lie algebra.

(b) Assume that L is nilpotent. Show that there is a basis B of L such that, for any $x \in L$, the matrix of $\mathrm{ad}(x)$ with respect to B is upper triangular with zero on the diagonal. Deduce that $\kappa(x,y) = 0$ for all $x, y \in L$. Thus, we have the implication:

$$L \text{ nilpotent} \quad \Rightarrow \quad \kappa(x,y) = 0 \text{ for all } x, y \in L. \tag{$*$}$$

(c) Show that, in general, we cannot replace "$\Rightarrow$" by "$\Leftarrow$" in $(*)$. For this purpose, check that

$$L = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & ia & c \\ 0 & 0 & 0 \end{pmatrix} \Big| a, b, c \in \mathbb{C} \right\} \qquad (\text{where } i = \sqrt{-1})$$

is a Lie subalgebra of $gl_3(\mathbb{C})$. Show that L is not nilpotent, but $\kappa(x,y) = 0$ for all $x, y \in L$.

(*Hint for (b)*: By Lie's Theorem, since L is soluble, there is a basis of L such that the matrix of $\mathrm{ad}(x)$ is upper triangular for all $x \in L$. Then use the fact that L is nilpotent.)

3. We consider the subspace

$$L = \left\{ \begin{pmatrix} a & a & b+c \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix} \Big| a, b, c \in \mathbb{C} \right\} \subseteq gl_3(\mathbb{C}).$$

(a) Verify that L is a Lie subalgebra of $gl_3(\mathbb{C})$. Show that L is soluble but not nilpotent.

(b) Find a Cartan subalgebra H of L. (*Hint.* Try subspaces of dimension 1.)

(c) Determine the weight space decomposition $L = H \oplus \bigoplus_{\lambda \in \Phi} L_\lambda$, where Φ is a set of non-zero linear maps $\lambda\colon H \to \mathbb{C}$ such that $\lambda([H,H]) = 0$. What is the cardinality of Φ? What are the dimensions of L_λ for $\lambda \in \Phi$?

4. **(a)** Define the notion of a root system Φ and of a set of simple roots Π.

List the possibilities for $|\Phi|$, assuming that $|\Pi| = 2$.

(b) Let L be a finite-dimensional, semisimple Lie algebra over $\mathbb{C}$.

(i) State the formula which expresses $\dim L$ in terms of the root system of L.

(ii) Show that $\dim L$ cannot be equal to 4, 5 or 7.

(iii) Give examples where $\dim L$ equals 6 or 8.