UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination

MX4543 Introduction to Lie algebras

Tuesday 22 May 2007 (3 pm to 5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.
The questions are independent and can be treated in any order.
Please note that full explanations form an essential part of your solution.

1.

(a) (i) Define the notion of a nilpotent Lie algebra.

(ii) Define the notion of a soluble Lie algebra.

(b) Let L be a three-dimensional Lie algebra over a field k. Assume that L has a basis $\{x, y, z\}$ such that

$$[x, y] = 0, \qquad [x, z] = 0, \qquad [y, z] = y.$$

(i) Show that L is soluble but not nilpotent.

(ii) Compute the Gram matrix of the Killing form of L with respect to the basis $\{x, y, z\}$.

2. **(a)** Define the Jordan–Chevalley decomposition of a matrix $A \in M_n(\mathbb{C})$.

(b) Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 0 & -2 \\ 2 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{C}).$$

The characteristic polynomial of A is $\chi_A = \det(A - XI_3) = X^2(1 - X)$. (You do not need to show this.)

(i) Show that A^2 is diagonalisable.

(ii) Show that $A - A^2$ is nilpotent.

(iii) Use (i) and (ii) to find the Jordan–Chevalley decomposition of A.

3. **(a)** State Lie's Theorem concerning soluble Lie algebras.

(b) Let k be a field of characteristic 2, that is, we have $1+1=0$ in k. (For example, $k=\mathbb{Z}/2\mathbb{Z}$.) Consider the following Lie subalgebra of $gl_2(k)$:

$$L = \left\{ \begin{pmatrix} 0 & a \\ a & b \end{pmatrix} \;\middle|\; a, b \in k \right\}.$$

(You do not need to show that this is a subalgebra.)

(i) Find a basis $\{x, y\}$ of L such that $[x, y] = x$.

(ii) Show that L is soluble.

(iii) The inclusion $L \subseteq gl_2(k)$ defines a representation of L. Thus, we may regard $V = k^2$ as an L-module. Show that this L-module is irreducible. (*Hint.* Note that a proper submodule of V would be spanned by a non-zero vector $v \in k^2$ such that $x.v$ and $y.v$ are scalar multiples of v. Show that no such vector exists.)

(iv) By (ii) and (iii), L is a soluble Lie algebra which admits a two-dimensional irreducible representation. Is this a contradiction to Lie's Theorem?

4. We consider the following subalgebra of $sl_3(\mathbb{C})$:

$$L := \left\{ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}.$$

(You do not need to show that this is a subalgebra.) The following elements form a basis of L:

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $[x, y] = z$, $[x, z] = -y$, $[y, z] = x$. (You don't need to verify this.)

(a) Use the above formulas to show that L is non-abelian and simple (that is, the only ideals are $\{0\}$ and L).

(b) Let

$$h = 2iz = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{where} \qquad i = \sqrt{-1} \in \mathbb{C}.$$

Show that there exist $e, f \in L$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Deduce that $L \cong sl_2(\mathbb{C})$.