UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION

MX4541 Electromagnetism

Thursday 26 May 2005 (9 am—11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Particles of charge q are placed at $(-1, 0, 0)$ and $(1, 0, 0)$. A particle of charge p is placed at $(0, 0, 1)$. The electric field due to these three particles vanishes at $(0, 0, -1)$. What is the relationship between p and q?

(b) Charge is distributed on the z-axis between $z = -L$ and $z = L$ with density $\rho(z)$, where $\rho(z) = \rho(-z)$ $(-L \leqslant z \leqslant L)$. Show that the resulting electric field at the point $(a, 0, 0)$ is

$$\mathbf{E}(a, 0, 0) = f(a)(1, 0, 0) \qquad \text{where} \qquad f(a) = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{a\rho(z)}{(a^2 + z^2)^{3/2}}\, dz.$$

Write down the value of the electric field at any point of the (x, y)-plane other than the origin.

2. **(a)** A continuous distribution of charge produces an electric field $\mathbf{E}$. State the relationship between the total amount of charge within any bounded region R and the flux of $\mathbf{E}$ through the boundary of R.

Use this result to show that if the charge density ρ of this charge distribution in $\mathbb{R}^3$ only depends on distance from the origin then the electric field due to this charge is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{Q(r)}{r^3}\, \mathbf{x},$$

where $r = \|\mathbf{x}\|$ and $Q(r)$ is the total charge within the sphere $\|\mathbf{x}\| \leqslant r$.

Show that $Q(r) = 4\pi \int_0^r t^2 \rho(t)\, dt$.

Assume that ρ is a continuous function of r. Prove that, given any $R > 0$, there is an $A > 0$ such that $|Q(r)| < Ar^3$ for $r < R$. Prove that, in this case, $\|\mathbf{E}(\mathbf{x})\| \to 0$ as $\mathbf{x} \to \mathbf{0}$.

3. **(a)** What is the force exerted on a particle of charge q by a magnetic field $\mathbf{B}$?

Consider the magnetic field $\mathbf{B}(\mathbf{x}) = (0, 0, \alpha)$, where α is a positive constant. A particle of mass m and charge q is initially at the origin with velocity $\mathbf{v} = (u, 0, 0)$. Show that the particle proceeds to move in a circular path. Where is the centre of the path?

(b) Particles P_k ($k = 1, \ldots, N$) have electric charges q_k and are fixed at the points $\mathbf{a}_k$. Show that the total energy $\mathcal{E}$ of this system is given by

$$\mathcal{E} = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{\|\mathbf{a}_i - \mathbf{a}_j\|} \ .$$

4. **(a)** A Plane Wave in the direction of the unit vector $\mathbf{n}$ is a function of the form

$$\phi(\mathbf{x}, t) = f(\mathbf{x} \cdot \mathbf{n} - ct)$$

where c is a positive constant and f is twice differentiable and *not constant*.

Show that ϕ satisfies the wave equation

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0.$$

Explain carefully the reason for the terminology 'plane wave in direction $\mathbf{n}$'.

Prove that a function ϕ cannot be a plane wave both in direction $\mathbf{n}$ and in direction $-\mathbf{n}$.

Prove more generally that if $\mathbf{n}$ and $\mathbf{m}$ are different unit vectors then a plane wave in direction $\mathbf{n}$ cannot be a plane wave in direction $\mathbf{m}$. It may help you to consider $\nabla\phi$.

(b) Maxwell's equations *in vacuo* are

$$\operatorname{div} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}, \quad \operatorname{curl} \mathbf{B} = \frac{1}{c^2}\dot{\mathbf{E}}.$$

A solution to these equations is of the form $\mathbf{E}(\mathbf{x}, t) = f(\mathbf{x}, t)\mathbf{n}$ and $\mathbf{B}(\mathbf{x}, t) = g(\mathbf{x}, t)\mathbf{m}$, where $\mathbf{n}$ and $\mathbf{m}$ are fixed unit vectors.

Show that either $\mathbf{n}$ and $\mathbf{m}$ are perpendicular or else $\mathbf{E}$ and $\mathbf{B}$ are constant.