UNIVERSITY OF ABERDEEN **SESSION 2003–2004**

DEGREE EXAMINATION

MX4540 Knots

Tuesday 25 May 2004 (3pm to 5pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

A table of knots is provided. You may use the information from this table in any of the questions below, but you should state where and how you are using it.

1. **(a)** Define what it means to say that two knots K_1 and K_2 are ambient isotopic.

(b) Define the three Reidemeister moves on a knot diagram.

(c) State Reidemeister's Theorem.

(d) Define what it means to say that a knot diagram is p-colourable for some prime number $p \geq 3$.

(e) Prove that the property of being p-colourable is a knot invariant.

2. In attempting to enumerate all 6-crossing alternating knots, the following potential Dowker sequences have to be considered:

6	8	10	2	12	4
6	8	10	12	4	2
4	8	10	12	2	6
8	10	2	12	4	6
6	8	10	4	12	2

Show that the first four are different Dowker sequences corresponding to the same knot diagram. Deduce that from these sequences we obtain exactly one mirror image pair of 6-crossing knots which are distinct from all knots with fewer crossings.

3. **(a)** Let K be a knot or link diagram. Let $< K >$ denote its bracket polynomial in the variable A. State the three rules that this polynomial must satisfy. Determine how the bracket polynomial changes under Reidemeister move 1.

(b) Define the writhe of an oriented knot or link diagram. Determine how the writhe changes under Reidemeister move 1.

(c) Define the Jones polynomial of an oriented knot or link. Assuming, without proof, that the bracket polynomial and the writhe of an oriented knot or link diagram is invariant under planar isotopies and Reidemeister moves 2 and 3, prove that two oriented knots or links which are ambient isotopic have the same Jones polynomial.

(continued on next page)

(d) Determine the Jones polynomial for the oriented link shown in Figure 1 below.

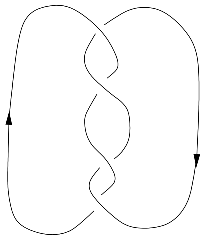


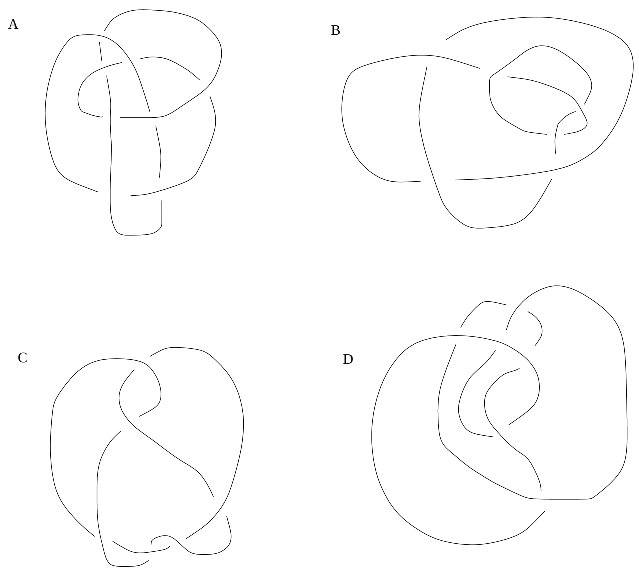
Figure 1:

(e) The four knots shown below have the following Jones polynomials:

$$v_1(t) = t^3 + t^5 - t^8, \quad v_2(t) = t^2 - t + 1 - t^{-1} + t^{-2},$$

$$v_3(t) = -t^3 + t^2 - t + 3 - t^{-1} + t^{-2} - t^{-3}, \quad v_4(t) = t^2 - t + 2 - 2t^{-1} + t^{-2} - t^{-3} + t^{-4}.$$

Identify which knot has which polynomial. [You should not need to work out the Jones polynomials for these knots].



4. **(a)** Let S be a path-connected subset of $\mathbf{R}^3$ and let $s_0 \in S$. Define the elements of the fundamental group $\Pi_1(S, s_0)$ and define multiplication in the group.

(b) Let S, S' be path-connected subsets of $\mathbf{R}^3$ and let $\phi : S \to S'$ be a homeomorphism, with $\phi(s_0) = s_0'$. Prove that ϕ induces a homomorphism

$$\phi^* : \Pi_1(S, s_0) \to \Pi_1(S', s_0').$$

Using ϕ^{-1} show that ϕ^* is an isomorphism.

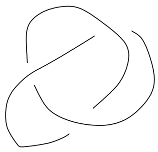
(c) Briefly explain how fundamental groups can be used to obtain invariants of knots and links.

(d) Using the link diagram shown in Figure 1 above, explain how to obtain generators and relations for the fundamental group of the complement of a knot or link.

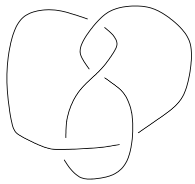
Determine the Wirtinger presentation for the link shown in Figure 1 above.

TABLE OF KNOTS

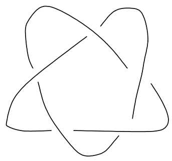
The table below gives all prime knots with crossing number no greater than 5, with one from each mirror image pair shown, their determinant, Jones polynomial and the bracket polynomial of the diagram shown.



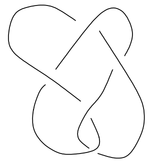
3_1



4_1



5_1



5_2

Knot	Determinant	Jones polynomial	Bracket polynomial
3_1	3	$t + t^3 - t^4$	$-A^5 - A^{-3} + A^{-7}$
4_1	5	$t^2 - t + 1 - t^{-1} + t^{-2}$	$A^8 - A^4 + 1 - A^{-4} + A^{-8}$
5_1	5	$-t^7 + t^6 - t^5 + t^4 + t^2$	$A^{-13} - A^{-9} + A^{-5} - A^{-1} - A^7$
5_2	7	$-t^6 + t^5 - t^4 + 2t^3 - t^2 + t$	$A^{-9} - A^{-5} + A^{-1} - 2A^3 + A^7 - A^{11}$