UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION
MX4536 Special Relativity
Thursday 27 May 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. Let I and I' be inertial frames in special relativity in standard configuration with relative speed v.

(i) Write down the Lorentz transformations from I to I' and from I' to I.

(ii) A particle has velocity $\mathbf{u} = (u_x, u_y, u_z)$ in I and velocity $\mathbf{u}' = (u'_x, u'_y, u'_z)$ in I'. Derive the velocity addition law in the form

$$u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}\,, \qquad u'_y = \frac{u_y}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}\,, \qquad u'_z = \frac{u_z}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}$$

where $\gamma = \left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$.

(iii) Write down equations similar to those in **(ii)** which give the components u_x, u_y and u_z in terms of u'_x, u'_y and u'_z and find the *speed* of a particle in I given that its *velocity* in I' is $(u, 0, 0)$ with $u > 0$.

(iv) A particle is observed to move in I from the point with space coordinates $(1, 1, 0)$ to the spatial origin with coordinates $(0, 0, 0)$ in a straight line with constant speed w. Find the velocity of the particle in I and I'.

(v) A photon travels along the x-axis with speed c as observed in I. Show that whether it moves in the positive or negative direction along the x-axis in I, its speed in I' is also c.

2. **(a)** A source of light O is at rest at the spatial origin of an inertial frame in special relativity. An observer O' moves with speed v along the positive direction of the x-axis of I as observed in I. If the frequency and wavelength of the light are ν and λ as measured by an observer at rest in I and ν' and λ' as measured by O' show that

$$\nu' = \left(\frac{c-v}{c+v}\right)^{\frac{1}{2}} \nu$$

and find a similar relationship between λ' and λ.

Comment briefly on the changes required to derive a similar result in the classical case.

(b) Define the *Lorentz inner product* $\langle\ \rangle$ as applied to 4-vectors in special relativity and use it to define the terms *spacelike*, *timelike*, *null* and *orthogonal.*

Let L, X and Y be 4-vectors satisfying $L \neq 0$, $\langle L, L\rangle = 0$, $\langle X, X\rangle = \langle Y, Y\rangle = 1$ and with L, X and Y mutually orthogonal. Let S be a set of 4-vectors defined by

$$S = \{\mu L + \nu X + \rho Y : \mu, \nu, \rho \in \mathbb{R}\}.$$

Establish the following results:

(i) Every non-zero member of S is orthogonal to L.

(ii) The only null members of S are of the form λL $(0 \neq \lambda \in \mathbb{R})$.

(iii) S contains no timelike member.

Show that if a non-zero 4-vector is orthogonal to a null vector then it is either a scalar multiple of that null vector or it is spacelike.

3. In special relativity a particle p has rest mass m_0 and moves with velocity $\mathbf{u}$ in an inertial frame I. Write down expressions for the particle's inertial mass m, momentum $\mathbf{p}$ and total energy E in I. Write down also expressions for the momentum and total energy of a photon in I if its frequency in I is ν. State the conservation laws which apply to collisions involving particles and photons.

By considering a particle originally at rest in I and which spontaneously disintegrates into two particles of equal rest mass and nothing else, show that rest mass may not be conserved in such an action.

A particle of rest mass m_0 is at rest in I. It emits a photon of frequency ν in I and as a result its rest mass is reduced to m_0' and its speed of recoil in I is u. Show that

$$u = \frac{h\nu c}{m_0 c^2 - h\nu}$$

and that

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = \frac{m_0 c^2 - h\nu}{(m_0^2 c^4 - 2m_0 c^2 h\nu)^{\frac{1}{2}}}.$$

Show also that the reduced rest mass m_0' satisfies

$$m_0' = \left[\frac{m_0}{c^2}(m_0 c^2 - 2h\nu)\right]^{\frac{1}{2}}.$$

4. Write short essays, including mathematical details where appropriate, on *two* of the following.

(i) The postulates of special relativity and their relation to the Michelson-Morley experiment.

(ii) The relativistic length contraction and time dilation phenomena.

(iii) The 4-dimensional formulation of relativistic mechanics and the ideas leading to the equation $E = mc^2$.