UNIVERSITY OF ABERDEEN **SESSION 2003–2004**

Degree Examination

MX4530 Time Series and Stochastic Processes

Monday 7 June 2004 (2pm to 4pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Six computational tasks execute one at a time. Execution starts in task 1 and tasks branch as follows:

Task 1 goes to task 3 and task 4 with equal probability,
If execution reaches task 2, then the program finishes,
Task 3 goes to task 2 and task 4 with equal probability,
Task 4 goes to task 3 and task 5 with equal probability,
Task 5 always goes to task 6, and
Task 6 always goes to task 5.

Considering the tasks to be the states of a discrete time Markov chain:

(i) Draw the Directed graph for this process, and write down the transition matrix for the Markov chain.

(ii) Find which states are periodic, and write down their periods.

(iii) Find the communicating classes, and label each as open or closed.

(iv) Calculate the probability of absorption in state 2 starting from state 1.

(b) A factory has two machines that make microchips that can either be good (G) or faulty (F). A technician notices that faulty microchips come in batches. He models the state of the machine as a discrete time Markov chain with two states: 1 – producing good microchips, and 2 – producing faulty microchips, with transition matrix

$$\mathbf{P} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}, \text{ where } 0 < \alpha, \beta < 1.$$

Show that the long run proportion of faulty chips is $\dfrac{1-\alpha}{2-\alpha-\beta}$.

(c) As a further investigation, the technician of part **(b)** observes 201 successive microchips for two machines **A** and **B**, obtaining the following data on transitions between states:

Machine **A**: $\mathbf{N} = \begin{bmatrix} 109 & 4 \\ 3 & 84 \end{bmatrix}$, Machine **B**: $\mathbf{N} = \begin{bmatrix} 75 & 7 \\ 7 & 111 \end{bmatrix}$.

Using the same model as in part **(b)**, test whether the transition matrix of machine **A** is different to that of machine **B**.

2. **(a)** Show that if $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent Poisson processes with rates λ_1 and λ_2 then $\{N_1(t) + N_2(t)\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

(b) Vehicles arrive at an inspection depot as a Poisson process with rate 10 per hour. On inspection, each vehicle has probability 0.99 of being passed as roadworthy, independently of any other.

(i) What is the probability that two or fewer vehicles arrive at the depot between 10am and 10.30am?

(ii) Given that there were 30 arrivals between 1pm and 2.30pm, what is the probability that there were 7 or fewer arrivals between 1.15pm and 1.45pm?

(iii) What is the expected number of failures in a four hour period?

(iv) What is the probability that all vehicles which arrive in an eight hour day are roadworthy?

(c) A computer program has four states:

1 Reading from disk,

2 Processing,

3 Successful completion and

4 Error.

The program starts in state **1** and takes an exponential length of time with mean 1 second before moving to state **2**.

In state **2** the program takes an exponential length of time with mean 1/2 second before jumping to state **3** with probability 0.5 or state **1** with probability 0.5.

Errors occur independently in states **1** and **2** as a Poisson process with a rate of 1 per 10 seconds, and when an error occurs, the program moves to state **4**.

Once in state **3** or state **4** the program finishes.

Considering the state of the program as a continuous time Markov chain, write down its generator matrix, and determine the transition matrix of the corresponding jump chain.

What is the probability that the program completes successfully?

3. A discrete state and discrete time unrestricted random walk $\{Z_n\}$ represents the total number of claims made at an insurance company up to time n, after n time periods from its starting point, 0. Each step X is an independent random variable with distribution:

$$\Pr(X = a) = \frac{e^{-\lambda}\lambda^a}{a!} \qquad \begin{array}{l} a = 0, 1, 2, \ldots, \\ \lambda > 0 \end{array}$$

and $Z_n = \sum_{i=1}^{n} X_i$.

(a) Determine the probability that Z_n will be in state k, given $Z_0 = 0$, and compute the probability for $\lambda = 0.1$ and $k = 6$ at time 20. (You may assume that Z_n is Poisson distributed with appropriate expectation.)

(b) Derive a 95% probability interval for Z_n, and compute the interval for $\lambda = 0.6$, $n = 1000$. Comment on the result, stating any assumptions made.

(c) Write brief notes on the following time series topics:

(i) cyclic behaviour

(ii) partial autocorrelation

(iii) the variogram and its uses.

4. **(a)** In the context of fitting an autoregressive model of order p (AR(p)) to a time series, explain what is meant by the Yule-Walker equations. Show that these equations lead to estimates

$$\hat{\phi}_1 = \frac{r_1(1 - r_2)}{1 - r_1^2} \qquad \hat{\phi}_2 = \frac{r_2 - r_1^2}{1 - r_1^2}$$

for the parameters of an AR(2) model, where r_1, r_2 are the first and second order sample autocorrelations.

(b) A series of length 100 has the following sample autocorrelations and partial autocorrelations for lags $1, 2, \ldots, 10$.

Lag k	1	2	3	4	5	6	7	8	9	10
r_k	0.49	-0.05	-0.27	-0.28	-0.07	0.22	0.20	-0.01	-0.09	-0.11
kth partial autocorrelation	0.49	-0.39	-0.06	0.01	-0.00	-0.02	0.00	0.01	0.03	0.00

Explain why it might be appropriate to fit an AR(2) model to this series.

(c) The mean of the 100 observations is 28.57 and the last two values (99th, 100th) are 26.74, 32.44 respectively. Fit an AR(1) model and an AR(2) model to the series and compare the forecasts you get for the 101st and 102nd terms in the series using the two models.