UNIVERSITY OF ABERDEEN **SESSION 2004-2005**

DEGREE EXAMINATION

MX4525 Inference 2

Tuesday 24 May 2005 (12 noon—2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. Consider the problem of estimating the unknown variance, σ^2, of a normally-distributed random variable X with mean zero. You are given a random sample of observations $X_1, X_2, \ldots, X_n$ of the variable X. Let $\theta = \sigma^2$.

(a) Let θ^* denote an unbiased estimator of θ. Find the Cramer-Rao lower bound for the variance of θ^*.

(b) Consider the unbiased estimator s^2, the sample variance. Determine whether s^2 is

(i) efficient;

(ii) asymptotically efficient.

[You are given that $\text{var}(s^2) = \dfrac{2\sigma^4}{(n-1)}$.]

(c) Consider the estimator $\tilde{\theta} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i^2$ of θ. Show that $\tilde{\theta}$ is

(i) unbiased and

(ii) efficient.

2. **(a)** Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric distribution with probability mass function

$$f(x|\theta) = \theta(1-\theta)^{x-1} \quad (x = 1, 2, 3, \ldots)$$

where θ is an unknown parameter $(0 < \theta < 1)$. Using standard results, find a complete sufficient statistic for θ.

(b) Let $X_1, X_2, \ldots, X_n$ be the results of a set of n independent Bernoulli trials, with $\Pr\{X_i = 1\} = \theta$, $\Pr\{X_i = 0\} = 1 - \theta$ for $1 \le i \le n$, where θ is an unknown parameter $(0 < \theta < 1)$.

(i) Show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ. (You may use standard results concerning the exponential family.)

(ii) Let $1 \le k \le n$, and define

$$h(T) = \frac{T(T-1)\cdots(T-k+1)}{n(n-1)\cdots(n-k+1)} \quad (0 \le T \le n).$$

[Note that $h(T) = 0$ for $T < k$.]

(1) Prove that $h(T)$ is an unbiased estimator of θ^k.

(2) Using standard results, show that $h(T)$ is a Minimum Variance Unbiased Estimator of θ^k.

3. **(a)** In the context of Bayesian estimation, define the following terms briefly:

(i) an improper prior; and

(ii) a Jeffreys prior.

(b) Let $x_1, \ldots, x_n$ denote a random sample from a normal distribution with mean μ and variance σ^2. It is assumed that the improper joint prior density for (μ, σ) is $\pi(\mu, \sigma) = \sigma^{-1}$, $-\infty < \mu < \infty$, $\sigma > 0$.

(i) Obtain an expression for the joint posterior density of (μ, σ), and show that it may be written in terms of the sample mean $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$ and sample variance $s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{(n-1)}$. (You need not evaluate the constant of proportionality.)

(ii) By integrating the posterior density with respect to σ, or otherwise, show that the marginal posterior density function for μ is proportional to

$$\{n(\bar{x} - \mu)^2 + (n-1)s^2\}^{-\frac{n}{2}}.$$

Hence show that the posterior distribution of $t = \dfrac{\sqrt{n}(\mu - \bar{x})}{s}$ follows Student's t-distribution, and state the number of degrees of freedom.

4. **(a)** In the context of decision theory, give formulae for the following loss functions:

(i) the quadratic loss function; and

(ii) the absolute error loss function.

(b) Suppose that $X \sim \text{Poisson}(\theta)$, where θ is unknown ($\theta > 0$).

(i) Find the risk function of the decision rule $\delta(x) = x$ using the quadratic loss function.

(ii) Suppose that the prior density is $\pi(\theta) = e^{-\theta}$ ($\theta > 0$). Calculate the Bayes risk of the rule $\delta(x) = x$ under quadratic loss.

(iii) Using a general result, find the Bayes estimate under quadratic loss when the prior density is $\pi(\theta) = e^{-\theta}$ ($\theta > 0$).

(iv) Let δ_1 denote the Bayes estimator under quadratic loss when the prior density is $\pi(\theta) = e^{-\theta}$ ($\theta > 0$). Calculate the Bayes risk of this rule under quadratic loss, and verify that the Bayes risk under quadratic loss of δ_1 does not exceed that of δ.

(Note: you are given that $\displaystyle\int_0^\infty \theta^2 e^{-\theta}\, d\theta = 2$.)