UNIVERSITY OF ABERDEEN **SESSION 2005-06**

Degree Examination
MX4507 Galois Theory
Wednesday 24 May 2006 (3 pm—5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. Decide which of the following complex numbers are constructible and give brief reasons for your answers.

(i) $\cos(\frac{2\pi}{5})$.

(ii) The primitive 7^{th} roots of 1 in $\mathbb{C}$.

(iii) The roots in $\mathbb{C}$ of the polynomial $x^3 + 3x - 1$.

(iv) The roots in $\mathbb{C}$ of the polynomial $x^3 - 2x - 1$.

(v) $\sqrt[8]{5}$.

2. Let ζ be a primitive 12^{th} root of unity in $\mathbb{C}$ and set $K = \mathbb{Q}(\zeta)$.

(i) Find the minimal polynomial of ζ over $\mathbb{Q}$.

(ii) Show that $\zeta^3 - 2\zeta$ is equal to one of $\pm\sqrt{3}$ and show that $i \in K$.

(iii) Determine the minimal polynomials of ζ over $\mathbb{Q}(i)$ and over $\mathbb{Q}(\sqrt{3})$.

(iv) Determine the structure of the Galois groups $\mathrm{Gal}(K : \mathbb{Q})$ and $\mathrm{Gal}(K : \mathbb{Q}(\sqrt{3}))$ and $\mathrm{Gal}(K : \mathbb{Q}(i))$.

(v) Determine the elements β in K whose minimal polynomials are solvable by radicals.

3. Let $\alpha = \sqrt{2 + \sqrt{2}}$ and set $K = \mathbb{Q}(\alpha)$.

(i) Find the minimal polynomial f of α over $\mathbb{Q}$.

(ii) Determine all roots in $\mathbb{C}$ of the minimal polynomial f of α.

(iii) Show that the field extension $K : \mathbb{Q}$ is normal.

(iv) Find the order and the structure of the Galois group $\mathrm{Gal}(K : \mathbb{Q})$.

(v) For each subgroup of $\mathrm{Gal}(K : \mathbb{Q})$ determine the corresponding subfield of K through the Galois correspondence.

4. Let p, q be two different prime numbers and set $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$.

(i) Find all constructible numbers in K.

(ii) Determine the degree $[K : \mathbb{Q}]$ and find an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.

(iii) Find a polynomial $f \in \mathbb{Q}[x]$ such that K is the splitting field of $\mathbb{Q}$ in $\mathbb{C}$.

(iv) Determine the structure of the Galois group $\mathrm{Gal}(K : \mathbb{Q})$.

(v) Find all subfields of K.