## Degree Examination

MX4505 Chaos and Fractals
Monday 24 May 2004
(9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination. Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. (a) Consider the fractal curve $\gamma:[0,1] \longrightarrow \mathbb{R}^{2}$ with generator

$$
x_{0}=(0,0), \quad x_{1}=\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right), \quad x_{2}=\left(\frac{1}{2}, 0\right), \quad x_{3}=(1,0) .
$$

(i) Sketch $\gamma_{0}, \gamma_{1}$ and $\gamma_{2}$.
(ii) Find orientation-preserving similarities $f_{1}, f_{2}, f_{3}: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ mapping $(0,0)$ to $x_{i-1}$ and $(1,0)$ to $x_{i}$, where $1 \leq i \leq 3$. Describe $f_{1}, f_{2}, f_{3}$ geometrically in terms of scaling, rotation, and translation.
(iii) Find the similarity dimension of $\gamma$ (with respect to the iterated function system $\left.f_{1}, f_{2}, f_{3}\right)$.
(iv) Recall that the path distance between two paths $\alpha, \beta:[0,1] \longrightarrow \mathbb{R}^{2}$ is defined by

$$
d(\alpha, \beta)=\max \{d(\alpha(t), \beta(t)) \mid t \in[0,1]\}
$$

For the path distance $d\left(\gamma_{0}, \gamma_{1}\right)$, find the time $t_{0} \in[0,1]$ at which the maximum distance occurs.
(v) Find the Hausdorff distance between $\gamma_{0}$ and $\gamma_{1}$.
(b) (i) Let $X$ be a metric space. Define what is meant when a function $f: X \longrightarrow X$ is said to be a similarity of scale factor $\lambda$.
(ii) Suppose $f: X \longrightarrow X$ is a similarity of scale factor $\lambda$. Let $k$ be a nonnegative integer. Prove that $f^{k}$ is a similarity of scale factor $\lambda^{k}$.
2. Three orientation-preserving similarities $f_{1}, f_{2}, f_{3}: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$, are defined by $f_{i}(x)=\frac{1}{2} x+b_{i}$, where

$$
b_{1}=(0,0), \quad b_{2}=\left(\frac{1}{2}, 0\right), \quad b_{3}=\left(0, \frac{1}{2}\right) .
$$

Let $K\left(\mathbb{R}^{2}\right)$ be the collection of non-empty, compact subsets of $\mathbb{R}^{2}$. Define $F: K\left(\mathbb{R}^{2}\right) \longrightarrow$ $K\left(\mathbb{R}^{2}\right)$ by $F(A)=f_{1}(A) \cup f_{2}(A) \cup f_{3}(A)$. You may assume that

$$
d(F(A), F(B)) \leq \frac{1}{2} d(A, B)
$$

for any $A, B \in K\left(\mathbb{R}^{2}\right)$. Let $M_{0}$ be the closed square

$$
M_{0}=\left\{(x, y) \in \mathbb{R}^{2} \mid 0 \leq x \leq 1,0 \leq y \leq 1\right\}
$$

Let $M_{n}=F^{n}\left(M_{0}\right)$.
(i) Sketch $M_{0}, M_{1}, M_{2}$.
(ii) For each $n \in \mathbb{N}$, show that $d\left(M_{n-1}, M_{n}\right) \leq \frac{1}{2^{n-1}} d\left(M_{0}, M_{1}\right)$.
(iii) Let $m, n \in \mathbb{N}$ and assume that $m>n$. Show that $d\left(M_{n}, M_{m}\right) \leq \frac{1}{2^{n-1}} d\left(M_{0}, M_{1}\right)$. Deduce that the sequence $\left(M_{n}\right)$ is a Cauchy sequence.
(iv) Since $K\left(\mathbb{R}^{2}\right)$ is a complete metric space, the Cauchy sequence $\left(M_{n}\right)$ converges to a set $M \in K\left(\mathbb{R}^{2}\right)$. Show that $M$ is a fixed point of $F$. That is, show that $F(M)=M$.
(v) Prove that $M$ is the unique fixed point of $F$.
(vi) Calculate the box dimension of $M$.
3. (a) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous map. Suppose $a \in \mathbb{R}$ is a fixed point of $f$.
(i) What is meant by saying that $a$ is an attracting fixed point?
(ii) What is meant by saying that $a$ is a repelling fixed point?
(iii) If $f$ is also differentiable, state a criterion which determines when a hyperbolic fixed point of $f$ is attracting or repelling.
(b) Suppose $f, g: \mathbb{R} \longrightarrow \mathbb{R}$ are two differentiable maps and $a \in \mathbb{R}$ is a hyperbolic fixed point of both $f$ and $g$ which is repelling for both $f$ and $g$. Prove that $a$ is also a repelling fixed point of $f \circ g$.
(c) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be the map defined by $f(x)=x^{2}+2 x-2$.
(i) Find the hyperbolic fixed points of $f$. Determine whether they are attracting or repelling.
(ii) Find the hyperbolic period 2 orbit of $f$. Determine whether it is attracting or repelling.
4. (a) Let $f, g: \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous maps. What is meant by saying that $f$ and $g$ are topologically conjugate?
(b) Let $f, g: \mathbb{R} \longrightarrow \mathbb{R}$ be the logistic maps $f(x)=\frac{2}{3} x(1-x)$ and $g(x)=\frac{4}{3} x(1-x)$. Find constants $\alpha$ and $\beta$ such that $f$ and $g$ are topologically conjugate by a homeomorphism of the form $\phi(x)=\alpha x+\beta$.
(c) Let $f, g: \mathbb{R} \longrightarrow \mathbb{R}$ be topologically conjugate by a homeomorphism $\phi: \mathbb{R} \longrightarrow \mathbb{R}$. Prove there is a one-to-one correspondence betwen the fixed points of $f$ and the fixed points of $g$.
(d) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be the tent map

$$
f(x)= \begin{cases}2 x & \text { for } x \leq \frac{1}{2} \\ 2(1-x) & \text { for } x \geq \frac{1}{2}\end{cases}
$$

(i) Find an explicit formula for $f^{2}$. Sketch the graphs of $f$ and $f^{2}$.
(ii) Let $x_{1} \in[0,1]$ be a point whose orbit does not contain the point $x=\frac{1}{2}$. Find the Lyapunov exponent $h\left(x_{1}\right)$.

