UNIVERSITY OF ABERDEEN **SESSION 2006-07**

DEGREE EXAMINATION

MX4082 Galois Theory

Thursday 18 January 2007 (12 noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions in complete grammatical sentences. All questions carry equal weight. Within each question, all sub-parts carry equal weight.

1. Decide which of the following complex numbers are constructible and give brief reasons for your answers.

(a) $\sin(\frac{2\pi}{5})$.

(b) $\sqrt{2+\sqrt[3]{2}}$.

(c) The roots in $\mathbb{C}$ of the polynomial x^3+5x-1.

(d) The roots in $\mathbb{C}$ of the polynomial x^3-2x-1.

(e) $\sqrt[8]{p}$, where p is a prime number.

2. Let $\zeta = e^{\frac{\pi i}{4}}$, so that ζ is a primitive 8-th root of unity in $\mathbb{C}$. Set $K = \mathbb{Q}(\zeta)$.

(a) Find the minimal polynomial of ζ over $\mathbb{Q}$.

(b) Show that $\zeta+\zeta^{-1}=\sqrt{2}$ and that $i \in K$.

(c) Determine the minimal polynomials of ζ over over $\mathbb{Q}(i)$ and over $\mathbb{Q}(\sqrt{2})$.

(d) Determine the order and structure of the Galois group $\mathrm{Gal}(K:\mathbb{Q})$.

(e) For each subgroup of $\mathrm{Gal}(K:\mathbb{Q})$ determine the corresponding subfield of K through the Galois correspondence.

3. Let $\alpha = \sqrt{2+\sqrt{2}}$ and set $K = \mathbb{Q}(\alpha)$.

(a) Find the minimal polynomial f of α over $\mathbb{Q}$.

(b) Determine all roots in $\mathbb{C}$ of the minimal polynomial f of α.

(c) Show that the field extension $K:\mathbb{Q}$ is normal.

(d) Find the order and the structure of the Galois group $\mathrm{Gal}(K:\mathbb{Q})$.

(e) Determine the elements β in K whose minimal polynomials are solvable by radicals.

4. Let ζ be a primitive cube root of unity in $\mathbb{C}$ and let $\sqrt[3]{2}$ be the real cube root of 2. Set $K = \mathbb{Q}(\zeta, \sqrt[3]{2})$.

(a) Determine the minimal polynomials of ζ and $\sqrt[3]{2}$ over $\mathbb{Q}$.

(b) Determine the degree $[K : \mathbb{Q}]$.

(c) Find a polynomial $f \in \mathbb{Q}[x]$ such that K is the splitting field of f in $\mathbb{C}$.

(d) Determine the order of the Galois group $\mathrm{Gal}(K : \mathbb{Q})$ and decide whether $\mathrm{Gal}(K : \mathbb{Q})$ is abelian or not.

(e) Find all constructible numbers in K.