DEGREE EXAMINATION

MX4034 Electromagnetism

Friday 23 January 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** State Coulomb's law for the electrostatic force between two charged particles. Particles A and B, both of charge q, are placed at $(a, 0, 0)$ and $(-a, 0, 0)$ respectively. Write down the electric field produced by A and B and find the magnitude of the force that they exert on a particle of charge $-2q$ at the point $(0, 0, a)$.

(b) A positive charge Q is distributed uniformly in a 1-dimensional distribution along the z-axis on the interval $-L \leqslant z \leqslant L$. Let $\mathbf{E}$ be the electric field due to this charge distribution.

Use symmetry considerations to determine the directions of $\mathbf{E}(L, 0, 0)$ and $\mathbf{E}(0, 0, 2L)$.

Show that if $z > L$ then $\mathbf{E}(0, 0, z) = \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{Q}{z^2 - L^2}\right)(0, 0, 1)$.

Explain briefly why $\mathbf{E}(0, 0, z)$ is not defined for $-L \leqslant z \leqslant L$.

2. **(a)** A particle P of charge q is placed at the point $\mathbf{a} \in \mathbb{R}^3$. Let Σ be the surface of the sphere with centre at $\mathbf{a}$ and radius r. Write down the electric field $\mathbf{E}$ due to the charge P and calculate its flux through Σ if the unit normals to Σ are taken to point away from the centre. Use this to prove that if R is any region containing P then the flux of $\mathbf{E}$ through the boundary of R is q/ϵ_0. (You may assume that $\operatorname{div} \mathbf{E}(\mathbf{x}) = 0$ when $\mathbf{x} \neq \mathbf{a}$.)

(b) A fixed system of charges consists of a charge $2q$ at the origin and charges $-q$ at the points $\pm\mathbf{a} = (0, 0, \pm a)$. Show that the leading term in the multipole expansion of the potential $V(\mathbf{x})$ is

$$\frac{1}{4\pi\epsilon_0}\frac{qa^2}{r^3}(1 - 3\cos^2\theta)$$

where $r = \|\mathbf{x}\|$ and θ is the angle between $\mathbf{x}$ and $\mathbf{a}$.

3. **(a)** Use the law of conservation of charge to derive the equation of continuity in the form

$$\operatorname{div}\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

for a charge distribution with density ρ and current density $\mathbf{J}$.

(b) Let $\mathcal{R}$ be a bounded region in $\mathbb{R}^3$ and f a real-valued function defined on the boundary $\partial\mathcal{R}$ of $\mathcal{R}$. Show that there cannot be more than one function ϕ defined on $\mathcal{R}$ such that $\nabla^2\phi = 0$ on $\mathcal{R}$ and $\phi = f$ on $\partial\mathcal{R}$.

(You may assume the Green's identity $\int_{\mathcal{R}} \|\nabla\phi\|^2 + \phi\,\nabla^2\phi\,dV = \int_{\partial\mathcal{R}} \phi\nabla\phi.\mathbf{N}\,dA$.)

4. **(a)** Let $c > 0$. Write down the general solution to the wave equation $u_{xx} - \frac{1}{c^2}u_{tt} = 0$ in $\mathbb{R}^2$.

Suppose that $u(x,0) = a(x)$ and $u_t(x,0) = 0$ for all $x \in \mathbb{R}$. Show that

$$u(x,t) = \frac{1}{2}(a(x+ct) + a(x-ct))$$

(b) Maxwell's equations *in vacuo* are

$$\operatorname{div}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{B} = 0, \qquad \operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}, \quad \operatorname{curl}\mathbf{B} = \frac{1}{c^2}\dot{\mathbf{E}}.$$

Find all solutions that are of the form

$$\mathbf{E}(x,y,z,t) = (a(u), b(u), 0), \qquad \mathbf{B}(x,y,z,t) = (p(u), q(u), r(u))$$

where $u = x - ct$.