UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION

MX4033 Number Theory

Thursday 22 January 2004 (3pm to 5pm)

Answer THREE questions. All questions carry equal weight.

1. **(a)** Calculate $\left(\frac{-2}{13}\right)$.

Hence, or otherwise, show that $(x^2+2)(x+4) \equiv 0 \pmod{13}$ has exactly one solution mod 13.

Solve the congruence $x^3+4x^2+2x+8 \equiv 0 \pmod{1521}$.

(b) For which primes p does the congruence $5x^2+6x+2 \equiv 0 \pmod{p}$ have at least one solution?

2. **(a)** Calculate $\left(\frac{273}{809}\right)$ and $\left(\frac{267}{811}\right)$.

(b) State Gauss's Law of Quadratic Reciprocity in the form

$$\left(\frac{p}{q}\right) = (-1)^k \left(\frac{q}{p}\right) .$$

(State the conditions on p and q and the expression for k.)

Use this this to show that if p and q are odd primes such that $p = q + 4a$ for some odd integer a, then $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$.

3. **(a)** Show that if p is both a quadratic residue mod 3 and congruent to 1 mod 4, then p is congruent to 1 mod 12. Then find the smallest prime p satisfying both of the following conditions:

(i) $p \equiv 1 \pmod{4}$;

(ii) 3, 5 and 7 are all quadratic residues mod p.

(b) By considering f^2+4 for a suitable integer f, show that there are an infinite number of primes of the form $12k+5$.

4. **(a)** Let d be a square-free integer congruent to 3 mod 4, let $\alpha = a + b\sqrt{d}$ be an element in the field $\mathbb{Q}(\sqrt{d})$ and let $\bar{\alpha} = a - b\sqrt{d}$ be the algebraic conjugate of α. Show that if both $\alpha\bar{\alpha}$ and $\alpha + \bar{\alpha}$ are integers, then both a and b are integers.

(b) Show that there is no element, α, in I_{-17} with the property that $\alpha^2 = 18$.

Show that there is an ideal, P, in I_{-17} such that $P^2 = \langle 18 \rangle$.