UNIVERSITY OF ABERDEEN **SESSION 2005-06**

Degree Examination

MX4008 Topology

Thursday 19 January 2006 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define what is meant by the **closure** of a subset A of a metric space X.

(b) Let A be a connected subspace of a metric space X. Show that the closure of A in X, $\overline{A} = \mathrm{Cl}_X(A)$, is a connected subspace of X. You may use the fact that the closure of A in $\overline{A}$ is equal to $\overline{A}$, *i.e.* $\mathrm{Cl}_{\overline{A}}(A) = \overline{A}$, for any subset A of X.

(c) Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$ be a locally constant function, that is, for every $x \in X$ there exists some $\delta > 0$ such that $f(x') = f(x)$ for all $x' \in B_0^X(x, \delta)$.

Show that f is continuous. Show that if in addition X is connected then f is constant, namely $f(x) = f(x')$ for all $x, x' \in X$.

2. **(a)** Define what is meant by an **open cover** of a metric space X. What is a **Lebesgue number** of an open cover? What does Lebesgue's lemma say about open covers of compact metric spaces?

(b) Let (X, d) and (Y, ρ) be metric spaces and let $f{:}X \to Y$ be a continuous function. Prove that if X is compact then f is uniformly continuous.

(c) Let X denote the open interval $(0, \frac{3}{4})$ equipped with the usual metric. For every $n \geq 1$ consider the open sub-intervals $U_n = (\dfrac{1}{2^{n+1}}, \dfrac{3}{2^{n+1}})$. Regard U_n as an open ball $B_0(x_n, r_n)$ in X. What is x_n ? What is r_n ? Consider the open cover $\mathcal{U} = \{U_n\}_{n \geq 1}$ of X (you are not required to prove that $\mathcal{U}$ covers X). Show that for every $n \geq 1$, the point x_n belongs to U_n but not to any other U_k where $k \neq n$. Prove that $\mathcal{U}$ does not admit a Lebesgue number.

3. **(a)** What is meant by saying that a subset A of a metric space X is **nowhere dense**? What is a **complete** metric space? State Baire's Theorem.

(b) Deduce from Baire's theorem that if a non empty complete metric space is presented as the union $F_1 \cup F_2 \cup \ldots$ of closed subsets, then F_n contains an open ball for some $n \geq 1$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has derivatives of all orders, namely the functions $f', f'', f^{(3)}, f^{(4)}, \ldots$ are differentiable (hence continuous). Assume that for every $t \in \mathbb{R}$ there exists some $n \geq 1$ such that $f^{(n)}(t) = 0$. Define for every $k \geq 1$

$$F_k = \{t \in \mathbb{R} \ : \ f^{(k)}(t) = 0\}.$$

Show that F_k are closed subsets of $\mathbb{R}$ for all $k \geq 1$. What can you say about $\cup_{k\geq 1} F_k$? Use part (b) and the fundamental theorem of calculus to prove that f coincides with a polynomial function on some interval $[a, b]$ where $a < b$.

(d) Let X be a complete metric space. Consider a chain of inclusions of closed balls in X,

$$B(x_1, r_1) \supseteq B(x_2, r_2) \supseteq B(x_3, r_3) \supseteq \cdots$$

where $r_n \leq \dfrac{1}{2^n}$. Show that the sequence of the centres of these balls $(x_n)_{n=1}^{\infty}$ is convergent in X. Prove that its limit y belongs to each one of the balls $B(x_n, r_n)$.

4. **(a)** Define what is meant by a **loop** α based at x_0 in a metric space X. Define what is meant by saying that a loop α is **homotopic** to a loop β.

(b) Prove the Pasting Lemma: Let A and B be closed subsets of X such that $X = A \cup B$ and let $f{:}X \to Y$ be a function. If $f|_A$ and $f|_B$ are continuous then f is also continuous.

(c) Show that the relation of homotopy of loops is transitive, that is if α is homotopic to β and β is homotopic to γ then α is homotopic to γ.