UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION

MX4008 Topology

Tuesday 25 January 2005 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define when a function $f : X \to Y$ between metric spaces (X, d) and (Y, ρ) is continuous at a point x in X. What is meant by saying that a function f is continuous on X?

Consider the following three statements:

(i) f is continuous on X.

(ii) For every convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X whose limit is x, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$ in Y.

(iii) For every closed subset K of Y, its preimage $f^{-1}(K)$ is a closed subset of X.

Show that (i) implies (ii) and that (ii) implies (iii).

(b) Let $\mathcal{B}(\mathbb{N})$ be the set of all bounded functions $f : \mathbb{N} \to \mathbb{R}$, that is the functions with the property that

$$\sup_{k \in \mathbb{N}} |f(k)| < \infty.$$

We equip $\mathcal{B}(\mathbb{N})$ with a metric d where for any two elements f, g of $\mathcal{B}(\mathbb{N})$ we set

$$d(f, g) = \sum_{k=0}^{\infty} \frac{|f(k) - g(k)|}{2^k}.$$

(You do *not* have to prove that this is a metric).

For every integer $n \geq 1$ let $\delta_n : \mathbb{N} \to \mathbb{R}$ denote the function

$$\delta_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Calculate $d(\delta_n, \mathbf{0})$ and show that the sequence $\{\delta_n\}_{n=1}^{\infty}$ converges to $\mathbf{0}$ where $\mathbf{0}$ denotes the constant function $k \mapsto 0$.

Use this to show that the function $\Phi : \mathcal{B}(\mathbb{N}) \to \mathbb{R}$ defined by

$$\Phi(f) = \sup_{k \in \mathbb{N}} f(k)$$

is not continuous.

2. **(a)** Define what an open partition of a metric space X is. When is X called connected?

(b) Let X be a connected metric space and $f : X \to \mathbb{R}$ a continuous function. Prove that if $f(x_1) < f(x_2)$ for some $x_1, x_2 \in X$, then for every c such that $f(x_1) < c < f(x_2)$ there exists some $x \in X$ such that $f(x) = c$.

In the remaining sections of this question we fix an integer $n \geq 2$ and a continuous function $f : [0, 1] \to \mathbb{R}$ which satisfies $f(0) = f(1)$.

(c) Consider the function $g : [0, \frac{n-1}{n}] \to \mathbb{R}$ defined by

$$g(t) = f(t) - f(t + \frac{1}{n}).$$

Explain why the functions $i, \varphi : [0, \frac{n-1}{n}] \to [0, 1]$, where $i(t) = t$ and $\varphi(t) = t + \frac{1}{n}$, are continuous. By considering the compositions of f with i and φ, explain why g is a continuous function.

(d) Show that $g(0) + g(\frac{1}{n}) + g(\frac{2}{n}) + \cdots + g(\frac{n-1}{n}) = 0$. Deduce that if $g(0) \neq 0$ then there exists some $k \in \{1, \ldots, n-1\}$ such that the sign of $g(\frac{k}{n})$ is opposite to the sign of $g(0)$.

(e) Use **(b)** and **(d)** to prove that there exists $0 \leq t_0 \leq \frac{n-1}{n}$ for which $g(t_0) = 0$. Deduce that $f(t_0) = f(t_0 + \frac{1}{n})$.

3. **(a)** Define what a Lebesgue number of an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of a metric space X is.

Prove that every open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of a compact metric space X has a Lebesgue number.

(b) Fix a compact metric space X and a continuous function $f : X \to X$.

Define a function $F : X \to \mathbb{R}$ by $F(x) = d(x, f(x))$. Prove that F is continuous. You may use, without proof, the fact that if $x_n \xrightarrow{n\to\infty} x$ and $y_n \xrightarrow{n\to\infty} y$ are convergent sequences in X, then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Explain what it means for F to attain its minimum, and explain why this happens. Deduce that if f does not have a fixed point, namely $f(x) \neq x$ for all $x \in X$, then there exists some $\epsilon > 0$ such that $d(x, f(x)) \geq \epsilon$ for all $x \in X$.

4. **(a)** Define what a G_δ subset of a metric space X is.

(b) Prove that the set $\mathbb{Q}$ of rational numbers is not a G_δ subset of $\mathbb{R}$.

(c) Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ whose set of points of continuity is exactly $\mathbb{Q}$? Justify your answer.

(d) Let the rational numbers be ordered in a sequence $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$ and define a function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = q_n \end{cases}$$

Using the fact that every open interval in $\mathbb{R}$ contains irrational numbers to prove that f is not continuous at any rational point $q \in \mathbb{Q}$.

Show that f is continuous at any irrational point $x \in \mathbb{R} \backslash \mathbb{Q}$.