**UNIVERSITY OF ABERDEEN** **SESSION 2003–04**

DEGREE EXAMINATION
MX4008 Topology
Monday 19 January 2004 (3pm to 5pm)

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*Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.*

*Marks may be deducted for answers that do not show clearly how the solution is reached.*

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*Answer ONE question from SECTION A and TWO questions from SECTION B. All questions carry equal weight.*

## SECTION A

**1.** Define the concept of *connectedness* of a topological space.

Prove that the closed interval $[a, b]$ equipped with the standard topology is a connected space.

**2.** Define *sequential compactness* of a metric space.

Prove that a compact metric space (namely a metric space which is compact as a topological space) is sequentially compact.

## SECTION B

**State clearly every theorem used.**

**3.** Let $(-\infty, r]$ denote the closed interval $\{t \in \mathbb{R} : t \leq r\}$.

(i) Show that $\mathcal{B} := \{(-\infty, r] : r \in \mathbb{R}\}$ is a topological base for a topology on $\mathbb{R}$ (which is different, of course, from the standard one, but you are not required to prove this fact).

(ii) Show that in the topology $\mathcal{T}$ generated by $\mathcal{B}$, if a subset $U$ of $\mathbb{R}$ is open then unless $U$ is empty or equal to $\mathbb{R}$, either $U = (-\infty, r]$ or $U = (-\infty, r)$ where $r = \sup U$.

(iii) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous where both the domain and range are equipped with the topology $\mathcal{T}$ above, then $f$ is increasing, namely $f(x) \leq f(y)$ whenever $x \leq y$.

(iv) Assuming that $\mathbb{R}$ is equipped with the topology $\mathcal{T}$, show that every subspace $Y$ of $\mathbb{R}$ (namely a subset $Y$ equipped with the subspace topology) is connected.

**4.** Let $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \,:\, x^2 + y^2 + z^2 + w^2 = 1\}$.

(i) Prove that $S^3$ is a closed subset of $\mathbb{R}^4$.

(ii) Let $S^3_+ = \{(x, y, z, w) \in S^3 \,:\, w \geq 0\}$ and let $S^3_- = \{(x, y, z, w) \in S^3 \,:\, w \leq 0\}$. Show that $S^3_+, S^3_-$ and $S^3$ are connected.

(iii) Prove that if $f : S^3 \to \mathbb{R}$ is a continuous function then there exist two *antipodal* points $u, v \in S^3$ such that $f(u) = f(v)$.

**5.** Let $X = C([0, 1], \mathbb{R})$ denote the set of all continuous functions $f : [0, 1] \to \mathbb{R}$ (both spaces are equipped with the standard topology).

(i) Show that

$$d(f, g) = \max\{|f(t) - g(t)| \,:\, t \in [0, 1]\}$$

is well defined, namely that the maximum is attained. Prove that $d$ is a metric on $X$.

(ii) Consider the sequence $f_n$ of functions ($n \geq 0$)

$$f_n(t) = \begin{cases} 1 - 2^n t & 0 \leq t \leq \frac{1}{2^n} \\ 0 & \frac{1}{2^n} \leq t \leq 1 \end{cases}$$

Sketch the graphs of $f_n$ and $f_{n+1}$ on the same axes system. Compute $d(f_n, 0)$ (where 0 is the constant function) for all $n \geq 0$ and show that $d(f_n, f_m) = 1 - 2^{n-m}$ for all $n < m$.

(iii) Let $\rho$ be the metric on $X$ defined by

$$\rho(f, g) = \int_0^1 |f(t) - g(t)| dt$$

(you are not required to show that $\rho$ is a metric). Calculate $\rho(f_n, 0)$. Prove that $\rho$ and $d$ induce different topologies on $X$.