UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination
MX4008 Topology
Wednesday 17 January 2007 (3 pm to 5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define what is meant by a *topological space* and a *basis* of a topological space.

(b) Let $\mathcal{T}$ be the following collection of subsets of the natural numbers $\mathbb{N}$:

$$\mathcal{T} = \{\emptyset, \mathbb{N}\} \cup \{U_n\}_{n=1}^{\infty}$$

where $U_n = \{1, 2, \ldots, n\}$.

(i) Show that $\mathcal{T}$ is a topology on $\mathbb{N}$.

(ii) Determine whether $\mathbb{N}$ with the topology $\mathcal{T}$ is connected.

(iii) Determine whether $\mathbb{N}$ with the topology $\mathcal{T}$ is Hausdorff.

(iv) Determine whether $\mathbb{N}$ with the topology $\mathcal{T}$ is compact.

(c) Suppose $f : X \longrightarrow Y$ is a function between topological spaces X and Y. Suppose the topology on Y has a basis $\mathcal{B}$. Show that f is continuous if and only if f has the property that $f^{-1}(B)$ is open whenever $B \in \mathcal{B}$.

2. **(a)** Define what is meant by a *metric space*. Define what is meant by a topological space X being *Hausdorff*.

(b) Regard a point $x \in \mathbb{R}^3$ in terms of its coordinates, $x = (x_1, x_2, x_3)$. Define a function $d : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

(i) Show that d is a metric on $\mathbb{R}^3$.

(ii) With respect to this metric, give an algebraic description or draw a picture of the open ball of radius 1 about the origin.

(c) Let M be a metric space. Prove that M is Hausdorff.

(d) Let X be a Hausdorff space and let x be a point in X. Show that the intersection of all the open sets containing X is $\{x\}$.

3. **(a)** Define what is meant by two topological spaces being *homeomorphic*. Define what is meant by a topological space being *connected*.

(b) Find an explicit homeomorphism between the closed intervals $[2, 4]$ and $[-1, 3]$. Justify that your map is a homeomorphism.

(c) Let X be a connected space and Y be a disconnected space. Prove that there is no continuous surjection $f : X \longrightarrow Y$.

(d) Show that the intervals $[1, 2)$ and $(3, 5)$ are not homeomorphic.

(e) Let X and Y be topological spaces. Define

$$t : X \times Y \longrightarrow Y \times X$$

by $t(x, y) = (y, x)$. Prove that t is a homeomorphism.

4. **(a)** Define an equivalence relation $\sim$ on $S^1 \times [0, 1]$ such that the quotient space

$$(S^1 \times [0, 1])/\sim$$

is homeomorphic to a cone. (It is not necessary to prove it is a homeomorphism.)

(b) Define an equivalence relation $\sim$ on $S^1 \times [0, 1]$ such that the quotient space

$$(S^1 \times [0, 1])/\sim$$

is homeomorphic to S^1. (It is not necessary to prove it is a homeomorphism.)

(c) Define an equivalence relation $\sim$ on $S^1 \times [0, 1]$ as follows. For every $x, y \in S^1$, let

$$(x, 0) \sim (y, 0), \qquad (x, \frac{1}{2}) \sim (y, \frac{1}{2}) \qquad \text{and} \qquad (x, 1) \sim (y, 1).$$

Let Q be the quotient space $Q = (S^1 \times [0, 1])/\sim$. Draw a picture of Q. To what well-known space is Q homeomorphic? (A proof is not necessary.)

(d) Define what is meant by a *continuous* map between topological spaces. Define what is meant by a topological space being *compact*.

(e) Let $f : X \longrightarrow Y$ be a continuous surjection between topological spaces X and Y. Show that if X is compact then Y is compact.

(f) Let X and Y be compact spaces. Suppose there is an equivalence relation $\sim$ on $X \times Y$. Let Q be the quotient space

$$Q = (X \times Y)/\sim .$$

Prove that Q is a compact space.