UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination
MX3531 Rings and Fields

Tuesday 22 May 2007 (12 noon to 2 pm)

Answer **ALL FOUR** *questions. All questions have equal weight.*

1. Let R be the ring of real valued continuous functions on $\mathbb{R}$, i.e. the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$, with addition and multiplication defined by $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$, respectively. Consider the evaluation map $\varphi : R \to \mathbb{R}$, defined by $f \mapsto f(0)$.

(a) Find the additive and multiplicative identity elements of R.

(b) Is R an integral domain?

(c) State the isomorphism theorem for rings.

(d) Show that φ is a ring homomorphism, **and** that φ is surjective.

(e) Show that $\ker(\varphi)$ is a maximal ideal of R.

2. Let $I = n\mathbb{Z}$ and $J = m\mathbb{Z}$. You may assume that the sets

$$I,\ J,\ IJ = \{ \sum_{i,j \text{ finite}} ij \mid i \in I,\ j \in J\} \quad \text{and} \quad I + J = \{i + j \mid i \in I,\ j \in J\}$$

are ideals of $\mathbb{Z}$.

(a) Show that $I + J = d\mathbb{Z}$, where $d = \gcd(n, m)$.

(b) Show that $IJ = nm\mathbb{Z} \subseteq I \cap J = l\mathbb{Z}$, where $l = \frac{nm}{d} \in \mathbb{Z}$ is the least common multiple of n and m.

(c) Deduce from parts **(a)** and **(b)** that $I + J = \mathbb{Z} \iff IJ = I \cap J \iff d = 1$.

3. **(a)** Consider the rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p$, for $p = 2, 3$ and 5. Factorise $t^3 - 2$ as a product of irreducible polynomials in $R[t]$, for each of the listed rings R.

(b) Show that the quotient ring $F = \mathbb{Q}[t]/(t^3 - 2)\mathbb{Q}[t]$ is a field and find the inverse of the class of $t + 1$ in F.

(c) Find $a, b \in \mathbb{R}[t]$ such that $af + bg = \gcd(f, g)$, for $f = t^6 + t^4 + 1$ and $g = t^2 - t + 1$.

4. **(a)** Give the definition of a principal ideal domain.

(b) Let p be a prime number and define $R_p = \{\frac{a}{b} \in \mathbb{Q} \mid \ p \text{ does not divide } b\}$.

(i) Find the characteristic and the group of units of R_p.

(ii) Show that any non zero ideal of R_p has the form $p^e R_p$, for some non-negative integer e.

(iii) Deduce from part (ii) that R_p is a principal ideal domain and that the set $pR_p = \{\frac{a}{b} \in R_p \mid \ p \text{ divides } a\}$ is the unique maximal ideal of R_p.

Identify the quotient ring R_p/pR_p, and find its characteristic.

(iv) Show that if p and q are distinct primes, then there are no nonzero ring homomorphisms between R_p and R_q.