UNIVERSITY OF ABERDEEN **SESSION 2005–06**

Degree Examination
MX3530 Applicable Mathematics
Friday 26 May 2006 (9 am to 12 noon)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer FIVE questions including no more than THREE from either section.
All questions carry equal weight.
Answer each section in a different answer book.

SECTION A (MX3526 Mathematical Methods)

1. **(a)** Let $f(x, y, z) = 2x + 3y^2 - zx$. What is ∇f? Find the rate of increase of f in the direction of the vector $(0, 1, 1)$ at the point $P = (1, 2, 1)$. Which unit vector at P points in the direction of maximum rate of increase of f?

(b) Let ϕ and $\mathbf{A}$ be a function and a vector field respectively, on $\mathbb{R}^3$. Assuming adequate differentiability, show that

$$\operatorname{div}(\operatorname{curl}\mathbf{A}) = 0 \qquad \text{and} \qquad \operatorname{div}(\phi\mathbf{A}) = \nabla\phi\cdot\mathbf{A} + \phi\operatorname{div}\mathbf{A}.$$

(c) Let $\mathbf{A}$ be the vector field on $\mathbb{R}^2$ given by $\mathbf{A}(x, y) = (x, y)$. Find the flow line of the field through the point $(1, 2)$.

2. **(a)** In $\mathbb{R}^2$, let γ be the curve that consists of straight line from $(-1, 1)$ to $(0, 1)$ followed by the circular arc $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$

(i) Produce parametrizations of each piece of γ.

(ii) Let $f(x, y) = x + y^2$. Evaluate $\int_\gamma f\, ds$.

(b) In $\mathbb{R}^3$, let γ be the path parametrised as $\mathbf{x}(t) = (\cos t, \sin t, t^3)$ for $0 \leqslant t \leqslant 2\pi$ oriented in the direction of increasing t. Let $\mathbf{A}$ be the vector field given by $\mathbf{A}(x, y, z) = (1, 1, z)$.

Evaluate the integral $\int_\gamma \mathbf{A}\cdot d\mathbf{s}$.

(c) Let γ be a smooth simple oriented curve in space starting at the point $(0, 0, 0)$ and ending at the point $(1, 1, 1)$. Let $\phi(x, y, z) = x^2 + y^2 + 2z$. What is the value of $\int_\gamma \nabla\phi\cdot d\mathbf{s}$? Give a brief explanation for your answer.

3. **(a)** Let Σ be the plane

$$z = x + 2y, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Let $f(x, y, z) = e^x + y$. Evaluate

$$\int_\Sigma f \, dA.$$

(b) A cylinder of radius 1 with axis along the z-axis can be parametrised as

$$\mathbf{x}(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \le \theta \le 2\pi$$

Find the outward-pointing unit normal to the cylinder at the point $\mathbf{x}(\theta, z)$. Evaluate the flux integral of the vector field $\mathbf{A} = (x, 0, y)$ through that portion of the surface that lies between $z = 0$ and $z = 1$.

(c) Let R be a bounded region of $\mathbb{R}^3$ and let $\mathbf{U}$ be a a vector field defined on R. Assume that the divergence theorem can be applied to R and $\mathbf{U}$ and that $\mathbf{U}$ has continuous second order partial derivatives. Let $\mathbf{N}$ be the outward-pointing unit normal to the boundary ∂R of R. Show that $\int_{\partial R} \mathrm{curl}\,(\mathbf{U}).\mathbf{N} \, dA = 0$.

4. **(a)** Let n be a positive integer. Find all solutions of the form $u(x, t) = e^{-n^2 t} f(x)$ to the one–dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

Which of these solutions satisfy the further condition that $u(0, t) = 0$ for all $t \ge 0$?

(b) Obtain the Fourier series for the function $f(x)$ which is periodic with period 2π and which is defined on $(-\pi, \pi]$ by $f(x) = x$. Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

SECTION B (MX3528 Optimisation Theory)

5. **(a)** What is meant by the Hessian matrix of a function $f\colon \mathbb{R}^n \to \mathbb{R}$? Under what conditions is it guaranteed to be symmetric?

What does it mean to say that a symmetric matrix is positive definite? What can you deduce about the nature of a critical point, if the Hessian matrix is positive definite there?

(b) Find the critical points of the function $f(x, y, z) = 2x^2 + y^2 + z^2 + 2xyz$. Write down the Hessian matrix and determine to what extent it tells you whether the critical points are local maxima, minima or saddle points.

6. **(a)** State the Karush–Kuhn–Tucker Theorem.

(b) Consider the problem of finding the maxima and minima of $f(x, y) = x^2 + y^2$ subject to the constraints $3x^2 + 4xy + 3y^2 \leqslant 270$ and $xy \geqslant 12$. Is the region described by these constraints compact? What can you deduce from this?

Use the Karush–Kuhn–Tucker Theorem to solve this optimisation problem.

7. **(a)** A pharmaceutical company is creating a tablet for a new drug. Each tablet is to contain a binder, a disintegrant and a filler, in addition to the active drug ingredient which is to be 14% of the weight of each tablet. Chemical and physical considerations mean that the weight of the disintegrant should not exceed 25% of the combined weights of the binder and the active ingredient, and that there should be at most 10 times as much filler as binder. The disintegrant costs £15/kg, the binder £50/kg and the filler costs £2/kg.

Express the problem of finding the proportions that minimise the cost of the tablets as a linear programming problem. [You are not asked to solve this linear programming problem.]

(b) Use the two phase simplex algorithm to find the maximal value of

$$f(x, y, z) = x + 2y + z$$

subject to the constraints

$$\begin{gathered} x \geqslant 0 \qquad y \geqslant 0 \qquad z \geqslant 0 \\ x + 2y + 2z \leqslant 9 \\ y + z \leqslant 3 \\ x + z \geqslant 4 \end{gathered}$$

8. **(a)** What is meant by a convex subset of $\mathbb{R}^n$? If C is a convex subset of $\mathbb{R}^n$, what is meant by a convex function $f\colon C \to \mathbb{R}$? If $f\colon C \to \mathbb{R}$ is convex, prove that every local minimum of f is a global minimum. Prove that the set of points at which f takes minimum value is a convex subset of C.

(b) Use the Hessian matrix to prove that

$$f(x, y) = 2x^2 + xy + y^2 - 9x - 4y - 5$$

is a convex function. Find the global minima for $f(x, y)$ on $\mathbb{R}^2$.