**UNIVERSITY OF ABERDEEN** **SESSION 2005–06**

DEGREE EXAMINATION

MX3529 Analysis and Algebra 2

Thursday 25 May 2006 (9 am to 12 noon)

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*Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.*

*Marks may be deducted for answers that do not show clearly how the solution is reached.*

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*Answer FIVE questions including no more than THREE from either section.*
*All questions carry equal weight.*
*Answer each section in a different answer book.*

## SECTION A (MX3502 Groups and Geometry)

**1.** **(a)** Let $G$ be a group and let $H$ be a subgroup of $G$. Define what is meant by a *left coset* of $H$ in $G$, and by a *right coset* of $H$ in $G$. Assuming that $G$ is finite, prove that $|H|$ divides $|G|$ (Lagrange's theorem).

**(b)** Define what is meant by the symmetric group $S_n$. Define what is meant by the sign $\epsilon(\lambda)$ of a permutation $\lambda \in S_n$. Define the alternating subgroup $A_n$ of $S_n$.

**(c)** The following are elements in $S_7$, written in cycle notation. Translate them into Cauchy notation.

$$\sigma = (234)(125)(3617), \qquad \lambda = (1234)(123)(12).$$

Find, and describe in cycle notation, $\sigma^{26}$ and $\lambda^{29}$. Find $\sigma\lambda$ and $\lambda\sigma$. Are they equal? Find the order $o(\sigma)$ of $\sigma$. Determine the orbits of $\lambda$. Find $\epsilon(\lambda)$, the sign of $\lambda$.

**(d)** Find a generator of $(\mathbb{Z}/19)^*$, the multiplicative group of nonzero elements of the field $\mathbb{Z}/19$. List all generators of $(\mathbb{Z}/19)^*$. List all elements of order 9 in $(\mathbb{Z}/19)^*$. Then list all subgroups.

**2.** **(a)** Define what is meant by an *action* of a group $G$ on a set $X$. Define what is meant by an *orbit* of the action. Define what is meant by the *stabiliser subgroup* $\mathrm{St}(y)$ for an element $y \in X$.

**(b)** Let $H$ be the set of all real $3 \times 3$–matrices which have the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

where $a_{11}, a_{22}, a_{33} \neq 0$. Explain why $H$ is a group under matrix multiplication.
There is a left action of $H$ on $\mathbb{R}^2$ defined in the standard manner by multiplying matrices with column vectors. (You are not required to verify that the conditions for an action are satisfied.) Determine the orbits of this action.

Select an element in each orbit. For each $\mathbf{v}$ that you have selected, find the stabiliser subgroup $\mathrm{St}(\mathbf{v})$.

**(c)** Suppose that a finite group $G$ acts on a set $X$. Let $y \in X$. State a formula which relates the number of elements in the orbit $Gy$ to the order of the stabiliser subgroup $\mathrm{St}(y)$.

Hence show that if a group $G$ of order 25 acts on a set $X$ with 79 elements, then the action has at least four fixed points. (A fixed point for the action is an element $z \in X$ such that $gz = z$ for all $g \in G$.) Also show that, if the same $G$ acts on a set with 20 elements, and there are no fixed points, then there will be 4 orbits.

**3.** **(a)** Let $G$ be a group with the neutral element $e$ and let $x \in G$. Define what is meant by the *order* $o(x)$ of $x$. Prove that $o(xy) = o(yx)$ for every $x, y \in G$.

**(b)** Define what is meant by a *subgroup* of a group. Define what is meant by a *cyclic group*. Show that every subgroup of a cyclic group is cyclic.

**(c)** Prove that every finitely generated subgroup of the additive group of rational numbers $(\mathbb{Q}, +)$ is cyclic. (Hint: use part (**b**)).

**(d)** Show that a group $G$ is abelian if and only if the map $f\colon G \to G$ given by

$$f(g) = g^{-1} \quad \text{for every } g \in G$$

is an automorphism of $G$.

**4.** **(a)** Let $F$ and $G$ be groups. Define what is meant by a homomorphism from $F$ to $G$. Define what is meant by the *kernel* and the *image* of such a homomorphism and what is meant by a *normal subgroup* of $F$.

Prove that the kernel of a homomorphism from $F$ to $G$ is a normal subgroup of $F$.

**(b)** Let $G$ be a group, and let $H$ be a normal subgroup of $G$. Explain how the set of left cosets $G/H$ can be made into a group. State the "first isomorphism theorem".

**(c)** Let $K$ be a normal subgroup of a group $G$ such that the quotient group $G/K$ has order $n$. Show that:

(i) $x^n \in K$ for every $x \in G$;

(ii) if $x \in G$ and $x^k \in K$ for some integer $k$ which is coprime to $n$, then $x \in K$.

**(d)** List all the homomorphisms from $\mathbb{Z}/7$ to $\mathbb{Z}/7$. Prove that your list is complete. Which of these homomorphisms are isomorphisms?

How many homomorphisms are there from the symmetric group $S_5$ to $\mathbb{Z}/13$? Justify your answer.

<u>SECTION B (MX3522 Complex Analysis)</u>

**5.** **(a)** Sketch the following subsets of the complex plane, giving a brief explanation of your results.

$$1 < |z - i| < 2, \qquad |z + i| = |\overline{z} + i|\,.$$

**(b)** Prove or provide a counterexample to each of the following statements (the Principal Value of the Logarithm is used).

$$\log(e^z) = z, \qquad e^{\log(z)} = z \qquad \forall z \in \mathbb{C} \setminus \{0\}.$$

**(c)** Evaluate $i^{2+3i}$, using the Principal Value for the power.

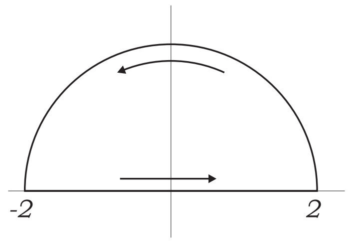
**(d)** Define what is meant by an open set in the complex plane. Prove that the following subset of $\mathbb{C}$ is open,

$$\mathcal{U} = \{x + iy \in \mathbb{C} : y > 0\}.$$

**6.** **(a)** Prove that the function $f(z) = \sin(\overline{z})$ is nowhere analytic.

**(b)** An entire function takes all its values in the set $\{t + it : t \in \mathbb{R}\}$. Prove that it is a constant function.

**(c)** Let $\gamma$ be the contour shown in the diagram. It consists of the real axis from $-2$ to 2, followed by the semicircle $|z| = 2$ from 2 to $-2$.
Evaluate the following integrals, stating carefully any results that you use,




$$\int_\gamma \overline{z}\,dz, \qquad \int_\gamma e^{z^2}\,dz, \qquad \int_\gamma \frac{e^z}{(z+i)(z-i)^2}\,dz.$$

**7.** **(a)** A function $f$ is analytic on and inside the contour $\gamma$ (traced anti-clockwise). State Cauchy's Integral Formula giving the value of the derivative $f^{(k)}(p)$ at a point $p$ inside $\gamma$.

Deduce that if $f$ is an entire function then, given any point $p \in \mathbb{C}$,

$$\left| f^{(k)}(p) \right| \leqslant \frac{k!M(r)}{r^k},$$

where $M(r)$ is the maximum value of $|f(z)|$ on the circle $|z - p| = r > 0$.

An entire function $f$ has the property that, for each $p \in \mathbb{C}$,

$$\left| \frac{f(z)}{(z-p)^2} \right| \to 0 \qquad \text{as} \qquad |z - p| \to \infty.$$

By considering the second derivative of $f$, or otherwise, prove that $f(z) = az + b$ for some constants $a$ and $b$.

**(b)** Let $f(z) = \dfrac{1}{(z+2)(z+i)}$. Find the radius of convergence of the Taylor series of $f$ expanded about the point $z = 1 + i$, stating clearly any result that you use.

Obtain the $z^{-3}$ term in the Laurent series of $f(z)$, expanded about $z = 0$, that is valid on the annulus $|z| > 2$.

**8.** **(a)** Each of the following functions has an isolated singularity at $z = 0$. In each case decide whether the singularity is removable, essential or is a pole. Find the residue at the singularity for each function.

$$\frac{e^{2z}}{z^2}, \qquad \frac{1-\cos z}{z^2}, \qquad \frac{1}{e^{1/z}}.$$

**(b)** Let $\gamma$ be the circle $|z - 1 - i| = 2$. Use the Residue Theorem to evaluate the integral

$$\int_\gamma \frac{1+2z^2}{1-z^4}\, dz.$$

**(c)** Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta} = \frac{2\pi}{\sqrt{5}}.$