UNIVERSITY OF ABERDEEN **SESSION 2004-05**

DEGREE EXAMINATION
MX3528 Optimisation Theory
Friday 27 May 2005 (9 am—11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** State the Karush-Kuhn-Tucker Theorem.

(b) Consider the following optimisation problem: find the minimum and maximum of $f(x, y, z) = xyz$ subject to the constraints $x^2 - y^2 + z^2 = 1$, $x^2 - z^2 = 1$, $x \geqslant 0$, $z \geqslant 0$ and $x - y + z \leqslant 1$. Write the three Karush-Kuhn-Tucker equations for this problem.

(c) Write all the systems of equations and inequalities resulting from the Karush-Kuhn-Tucker theorem, corresponding to two inequality constraints being active (strict), and the third inactive (lax). *Do not* attempt to solve the systems, although you may simplify them if you wish. (The number of equations must be the same as the number of variables.)

2. **(a)** Solve the optimisation problem:

$$\text{minimise} \quad x_1^2 + x_2^2 + x_3^2 \quad \text{subject to} \quad x_1 + x_2 + x_3 = 3 \quad \text{and} \quad x_1 - x_2 + x_3 = 5.$$

A fact that may become useful in your calculation is that the matrices

$$\begin{bmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 3 \\ 1 & -1 & 1 & 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

are row equivalent.

(b) Write down the Karush-Kuhn-Tucker equations for the problem:

$$\text{minimise} \quad x_1^2 + x_2^2 + x_3^2 \quad \text{subject to} \quad x_1 + x_2 + x_3 \geqslant 3 \quad \text{and} \quad x_1 - x_2 + x_3 \leqslant 5.$$

Find the global minimum.

3. **(a)** Define what it means for a subset $E \subseteq \mathbb{R}^n$ to be convex. Define also what it means for function $f\colon E \to \mathbb{R}$, defined on a convex subset $E \subseteq \mathbb{R}^n$, to be convex.

(b) Let $f\colon E \to \mathbb{R}$ be a convex function defined on a convex subset $E \subseteq \mathbb{R}^n$. Assume that f obtains a minimum on E and show that the set of all $\mathbf{x} \in E$, such that $f(\mathbf{x})$ is a local minimum of f on E, is a convex subset of $\mathbb{R}^n$, and that f is constant on it. (You may use the fact the any local minimum of a convex function is global.)

(c) Consider the functions

$$f(x, y) = x^2 + xy + y^2 - x - y - 1 \qquad \text{and} \qquad g(x, y) = x^2 + 4xy + 4y^2 + x + y.$$

Use the Hessian matrix to show that f is strictly convex and that g is convex over the entire xy-plane. Find the global extrema for f and g on the plane, if they exist.

4. **(a)** A highland craft industry produces handmade garments of two styles, using workers organised into three groups. Groups A, B and C are capable of 120, 110 and 70 hours work a week.

Each garment of style X requires 4, 2 and 1 hours work by groups A, B and C, respectively. Each garment of style Y requires 1, 3 and 2 hours work by groups A, B and C, respectively.

Suppose that the profit on each garment of style X is equal to three times the profit on each garment of style Y. The objective is to maximise the total profit. Formulate this problem as a linear optimisation problem in standard form. (You are not required to solve the problem.)

(b) Consider the linear optimisation problem:

maximise $6x_1 + 5x_2 + x_3$ subject to $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ and

$$\begin{aligned} x_1 + 2x_2 + 1x_3 &\leqslant 3, \\ x_1 \qquad\quad - x_3 &\leqslant 6, \\ x_2 + \ x_3 &\geqslant 1. \end{aligned}$$

Apply the simplex method to obtain the optimal value. Describe briefly the reasoning behind each step.