UNIVERSITY OF ABERDEEN **SESSION 2006-07**

DEGREE EXAMINATION
MX3528 Optimization Theory
Friday 25 May 2007 (3 pm to 5 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions

1. **(a)** Let $E \subseteq \mathbb{R}^n$ be a subset. Define what it means for E to be

i) an open subset,

ii) a closed subset, and

iii) a bounded subset.

Let $f\colon E \to \mathbb{R}$ be a continuous function, where E is a subset of $\mathbb{R}^n$. State a condition on E which ensures that f obtains a maximum and a minimum on E? You are not required to prove your statement.

(b) By using the Karush-Kuhn-Tucker method or otherwise, find the absolute minimum and maximum (if they exist) of the function $H(x, y, z) = x + y^2 + z$ on the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

2. **(a)** Let $E \subseteq \mathbb{R}^n$ be a subset. Define what it means for E to be a convex subset. Let E be a convex subset of $\mathbb{R}^n$, and let $f\colon E \to \mathbb{R}$ be a function. Define what it means for f to be a convex function.

(b) Let $f(x, y, z) = x^2+y^2+z^2$. By using the Hessian matrix of f, and stating the relevant theorem, show that f is a convex function on $\mathbb{R}^3$. Then, by using the Karush-Kuhn-Tucker method, find the absolute minimum of f, if it exists, subject to the constraints $1 \leqslant x^2 + 2y^2 - z^2 \leqslant 4$. Show that a maximum of f subject to this constraint does not exist.

3. **(a)** Consider the following optimisation problem: find the extrema of $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 + z^2 = 1$ and $xy + yz + zx = 1$. Use the method of Lagrange multipliers to write down a system of equations, the set of solutions of which contain all potential maxima and minima of f subject to the given constraints. Do not attempt to solve the system.

(b) Let $E \subseteq \mathbb{R}^n$ be a convex subset and let $f\colon E \to \mathbb{R}$ be a convex function. Show that for any $c \in \mathbb{R}$, the set

$$Z_f = \{\mathbf{x} \in E \mid f(\mathbf{x}) \leqslant c\}$$

is a convex subset of $\mathbb{R}^n$. Show furthermore that if $\{f_i\}_{i\in I}$ is any collection of convex functions on a convex subset $E \subseteq \mathbb{R}^n$, and $\{c_i\}_{i\in I}$ is a collection of real numbers, then the set

$$\{\mathbf{x} \in E \mid f_i(\mathbf{x}) \leqslant c_i,\ \forall i \in I\}$$

is a convex subset of $\mathbb{R}^n$.

4. **(a)** A Highland Malt distillery is producing malt whiskey of two different types, using workers organised into three groups. Groups A, B and C are capable of 7500, 6300 and 4200 hours' work a year.

Each 100 bottles of whiskey of type X require 400, 200 and 100 hours' work by groups A, B and C, respectively. Every 100 bottles of whiskey of type Y requires 100, 300 and 200 hours' work by groups A, B and C, respectively.

Suppose that the profit on each bottle of type X is equal to three times the profit on each bottle of type Y. The objective is to maximise the total profit. Formulate this problem as a linear optimisation problem in standard form. (You are not required to solve the problem)

(b) Solve the following linear optimisation problem:

Maximise $x_1 + 2x_2 + 3x_3 - 5$ subject to $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ and

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 4, \\ 2x_1 & + & 3x_2 & + & x_3 & \leq & 6, \\ -x_1 & & & - & x_3 & \geq & -3. \end{array}$$