UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION

MX3526 Mathematical Methods

Tuesday 31 May 2005 (12 noon—2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** A curve Γ in space has vector equation

$$\mathbf{x} = (\sin(\pi u),\, u^2 - 1, u^2 + 3u + 3), \qquad (u \in \mathbb{R}).$$

Find a vector equation of the tangent line to Γ at the point $(0,\, 0,\, 1)$.

(b) A surface Σ in space is has vector equation

$$\mathbf{x} = (u - v,\, u^3,\, v^3), \qquad ((u,\, v) \in \mathbb{R}^2).$$

Find a unit vector normal to Σ at the point $(0,\, 1,\, 1)$.

Let Q be any point of Σ other than the origin. Show that the line through Q and the origin either meets Σ in exactly one other point or else lies entirely in Σ.

(c) Find the rate of change of the scalar field $\rho = x^2 + 2y^2 + 3z^2 - xy - yz$ in the direction of the vector $(1,\, 1,\, 1)$ at the point $(1,\, 2,\, -1)$.

Find a coordinate equation of the tangent plane at the point $(1,\, 2,\, -1)$ to the level surface of ρ through that point.

2. **(a)** Let ϕ be a scalar field and let $\mathbf{v}$ be a vector field defined in a region of space. Show, assuming appropriate differentiability conditions, that

$$\operatorname{div}(\phi\mathbf{v}) = \mathbf{v} \cdot \operatorname{grad}\phi + \phi \operatorname{div}\mathbf{v}.$$

Suppose that $\psi = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ and f is a real valued function which is twice continuously differentiable on the interval $(0,\, \infty)$. Show that

$$\operatorname{grad}\psi = \frac{f'(r)}{r}\,\mathbf{x} \quad \text{and} \quad \nabla^2\psi = \frac{1}{r^2}\frac{d}{dr}\big(r^2 f'(r)\big) \qquad (\mathbf{x} = (x,\, y,\, z) \neq \mathbf{0}).$$

Deduce that if $\nabla^2\psi = 0$ (except possibly at the origin) then $\psi = A + \dfrac{B}{r}$ where A and B are constants.

(b) Let $\mathbf{v}$ be the vector field given by $\mathbf{v} = (yz + y + 2xz,\, xz + x + z^2,\, xy + x^2 + 2zy + 2z)$. Show that $\operatorname{curl}\mathbf{v} = \mathbf{0}$. Find a scalar field ψ such that $\operatorname{grad}\psi = \mathbf{v}$.

3. **(a)** Let Γ be a simple smooth oriented curve in space. Let $\mathbf{a}$ be the initial point and let $\mathbf{b}$ be the final point of Γ. Suppose that ϕ is a scalar field defined on a region containing Γ. Assuming appropriate differentiability conditions on ϕ, show that

$$\int_{\Gamma} \operatorname{grad} \phi \cdot d\mathbf{s} = \phi(\mathbf{b}) - \phi(\mathbf{a}).$$

(b) Let C be the region in space given by $0 \leqslant x,\, y,\, z \leqslant 1$ and let ∂C be the boundary of C oriented by the outward pointing unit normal. Suppose that $\mathbf{v}$ is the vector field given by $\mathbf{v} = (y^3 - 2xy,\, y^2 + 3y + 2zy,\, z - z^2)$. Evaluate $\displaystyle\int_{\partial C} \mathbf{v} \cdot d\mathbf{A}$, stating clearly any result used.

4. **(a)** Let f and g be real-valued functions, which are twice continuously differentiable on $\mathbb{R}$. Let $u(t,\, x) = f(x+t) + g(x-t)$. Show that $u = u(t,\, x)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad (-\infty < x < \infty, \quad -\infty < t < \infty).$$

Find the solution of this equation which satisfies the initial conditions

$$u(0,\, x) = e^{-x^2}, \quad \frac{\partial u}{\partial t}(0,\, x) = 0 \qquad (-\infty < x < \infty).$$

(b) Let f be the real-valued function defined on the interval $[-\pi, \pi]$ by

$$f(x) = \begin{cases} -1 & \text{if } -\pi \leqslant x \leqslant 0, \\ 1 & \text{if } \phantom{-\pi \leqslant{}} 0 < x \leqslant \pi. \end{cases}$$

Sketch the graph of this function.

Find the Fourier series of f on $[-\pi, \pi]$ and deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \; .$$