UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION

MX3526 Mathematical Methods

Tuesday 1 June 2004 (12noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Find the rate of increase of the function $f(x, y, z) = 2x^3 - y + 3z$ in the direction of the vector $(1, -1, 2)$ at the point $(2, 3, 0)$. Find also the unit vector in the direction of maximum rate of increase of f at the point $(1, 1, -1)$ and find this maximum rate of increase.

(b) Let ϕ and $\mathbf{A}$ be a differentiable function and a differentiable vector field, respectively, on $\mathbb{R}^3$. Show that

$$\operatorname{curl}(\nabla\phi) = \mathbf{0}$$

and

$$\operatorname{curl}(\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \operatorname{curl} \mathbf{A}.$$

Deduce that $\operatorname{curl}(\phi\nabla\phi) = \mathbf{0}$.

(c) Let $\mathbf{A}$ be the vector field on $\mathbb{R}^2$ given by $\mathbf{A} = (2x, -y)$. Find the flow line of $\mathbf{A}$ starting at the point $(1, 3)$.

2. **(a)** On that part of $\mathbb{R}^3$ with $z \geq 0$ let σ be the path $\mathbf{x}(t) = (\cos t, \sin t, t^2)$ $(0 \leqslant t \leq 2\pi)$ and f the function $f(x, y, z) = x^2 + y^2 + \sqrt{z} - 1$. Evaluate $\displaystyle\int_\sigma f\, ds$.

(b) In $\mathbb{R}^3$ let γ be the path $\mathbf{x}(t) = (e^t, t, \sin t)$ for $0 \leqslant t \leqslant 1$ and $\mathbf{A}$ the vector field given by $\mathbf{A} = (y, x, 0)$. Evaluate $\displaystyle\int_\gamma \mathbf{A} \cdot d\mathbf{s}$.

(c) On $\mathbb{R}^3$ let f be the function $f(x, y, z) = x^2 + 3z^2$ and let Δ be the two straight line paths joining the points $(1, 1, 1)$ to $(2, 1, 2)$ and $(2, 1, 2)$ to $(2, 2, 4)$. Find $\displaystyle\int_\Delta f\, ds$.

Find also an arc length parametrisation of the straight line from $(1, 1, 1)$ to $(2, 1, 2)$.

3. **(a)** A surface Σ in $\mathbb{R}^3$ is parametrised by $\mathbf{x}(u, v) = (u, v, u^2 + 2v^2)$. Find a unit normal to this surface at the point $(1, 1, 3)$. Write down an integral which gives the area of this surface lying between $z = 0$ and $z = 1$. Do not attempt to evaluate the integral.

(b) Write down a parametrisation for the cylinder whose axis is the z axis and which intersects the xy plane in the circle centred at the origin and of radius R.

Calculate the flux of the vector field $\mathbf{A} = (x, y, 0)$ over the surface of this cylinder between $z = 0$ and $z = 1$.

(c) State the divergence theorem and verify it for the vector field $\mathbf{A}$ given in **(b)** above and for the region enclosed by the cylinder in **(b)** and the planes $z = 0$ and $z = 1$.

4. **(a)** Let $f(x,t)$ satisfy the 1-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \tag{1}$$

where c is a positive constant. Let $u = x - ct$, $v = x + ct$ and let $F(u,v) = f\left(x(u,v), t(u,v)\right)$. Show that $\dfrac{\partial f}{\partial x} = \dfrac{\partial F}{\partial u} + \dfrac{\partial F}{\partial v}$ and find similar expressions for $\dfrac{\partial f}{\partial t}$, $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial t^2}$. Hence show that (1) is equivalent to the equation $\dfrac{\partial^2 F}{\partial u \partial v} = 0$ for F.

By writing this last equation as $\dfrac{\partial}{\partial u}\left(\dfrac{\partial F}{\partial v}\right) = 0$ find the general solution for F and hence show that the general solution of (1) is $f(x,t) = g(x - ct) + h(x + ct)$ where g and h are arbitrary functions of a single variable.

(b) Obtain the Fourier series of the function $f : \mathbb{R} \to \mathbb{R}$ which is periodic with period 2π and satisfies

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x \leq 0 \\ x & \text{for } \ 0 \leq x \leq \pi. \end{cases}$$