UNIVERSITY OF ABERDEEN **SESSION 2006–07**

Degree Examination

MX3526 Mathematical Methods

Thursday 24 May 2007 (12 noon to 2 pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Find the rate of change of the real-valued function $\rho = 2x^3 - y + 3z$ in the direction of the vector $(1, -1, 2)$ at the point $(2, 3, 0)$. Find the unit vector in the direction of the maximum rate of change of ρ at the point $(1, 1, -1)$ and give the maximum rate of change at this point.

(b) Let ϕ be a real-valued function and let $\mathbf{V}$ be a vector field defined in a region of space. Show, assuming appropriate differentiability conditions, that

$$\operatorname{curl} \nabla\phi = \mathbf{0}$$

and

$$\operatorname{curl}(\phi\, \mathbf{V}) = \nabla\phi \times \mathbf{V} + \phi \operatorname{curl} \mathbf{V}.$$

Deduce that $\operatorname{curl}(\phi\, \nabla\phi) = \mathbf{0}$.

(c) Let $\mathbf{A}$ be the vector field on $\mathbb{R}^2$ given by $\mathbf{A}(x, y) = (x, -y)$. Find the flow line of the field through the point $(3, 2)$.

2. **(a)** In $\mathbb{R}^2$, let γ be the curve that consists of the straight line from $(4, 0)$ to $(2, 0)$ followed by the circular arc $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$.

(i) Produce parametrizations of each piece of γ.

(ii) Let $f(x, y) = x^2 + y^2$. Evaluate $\int_\gamma f\, ds$.

(b) Let

$$\mathbf{A}(\mathbf{x}) = \frac{1}{x^2 + y^2}(y, -x, 0)$$

be the vector field which is defined everywhere apart from the z-axis. Let γ be the circle $(\cos t, \sin t, 0)$, $0 \le t \le 2\pi$ in the (x, y) plane.

(i) Evaluate $\displaystyle\int_\gamma \mathbf{A} \cdot d\mathbf{s}$.

(ii) Is there a function ϕ defined everywhere apart from the z-axis such that $\mathbf{A} = \nabla\phi$? Give a brief explanantion for your answer.

3. **(a)** Let Σ be the sphere of radius 1 and center $(0,0,0)$ in $\mathbb{R}^3$. Evaluate the integral $\int_\Sigma \sqrt{x^2+y^2}\, dA$.

(b) A cylinder of radius R with axis along the z-axis can be parametrised as

$$\mathbf{x}(\theta, z) = (R\cos\theta, R\sin\theta, z)$$

Find the outward-pointing unit normal to the cylinder at the point $\mathbf{x}(\theta, z)$. Evaluate the flux integral of the vector field $\mathbf{A} = (xz, 0, yz)$ through that portion of the surface that lies between $z = 0$ and $z = R$.

(c) Let R be a bounded region of $\mathbb{R}^3$ appropriate for the application of the Divergence theorem. Show that if $\mathbf{U}$ is a vector field defined on R then $\int_{\delta R} \operatorname{curl} \mathbf{U}.\mathbf{N}\, dA = 0$.

4. **(a)** Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, x \geq 0$$

which satisfies the initial conditions

$$u(0,x) = \sin x, \quad \frac{\partial u}{\partial t}(0,x) = e^x, \quad x \geq 0.$$

(b) Let f be the real-valued function which is periodic with period 2π and which is defined on $[-\pi, \pi]$ by $f(x) = x^2$. Sketch the graph of this function.

(i) Obtain the Fourier series of f.

(ii) Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots.$$