UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION

MX3522 Complex Analysis

Thursday 2 June (9 am—11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Sketch the following subsets of the complex plane, giving a brief explanation of your results.

$$|z+2| > 1, \qquad |z-i| < |z+1|\,.$$

(b) Show that the equation $\tan(z) = -i$ has no solutions.

(c) Evaluate $\left|(-i)^{2+i}\right|$, using the Principal Value for the power.

(d) Let p and q be two complex numbers such that $|p-q| > 2$. Let

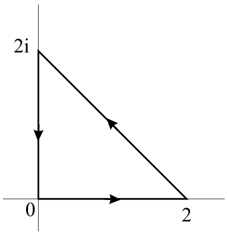
$$B_1 = \{z \in \mathbb{C} : |z-p| < 1\}, \qquad B_2 = \{z \in \mathbb{C} : |z-q| < 1\}.$$

Prove that $B_1 \cap B_2 = \emptyset$.

2. **(a)** Prove that the function $f(z) = z\bar{z}$ is nowhere analytic but that it is differentiable at one point.

(b) An entire function takes all its values on the circle $|z| = 1$. Prove that it is a constant function.

(c) Let γ be the triangular contour shown in the diagram, from 0 to 2 then to $2i$ and back to 0.
Evaluate the following integrals, stating carefully any results that you use,



$$\int_\gamma \overline{z}\,dz, \qquad \int_\gamma \sin(z)\,dz, \qquad \int_\gamma \frac{e^z}{(2z-1-i)^2}\,dz.$$

3. **(a)** State Cauchy's Integral Formula giving the value of a function f at a point p inside a contour γ on and inside of which f is analytic.

Deduce that, with this notation,

$$f'(p) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-p)^2}\,dz.$$

(b) Suppose that f is an entire function that takes real values on the real axis. Prove this is also true for the first derivative. Assuming that it holds for all higher derivatives, prove that $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

4. **(a)** Define what is meant by saying that an isolated singularity of the complex function f is a simple pole with residue -1.

(b) Find the coefficients of the z^3 and z^{-3} terms in the Laurent series of $\dfrac{1}{(z+1)(z-2)}$ expanded in the annulus $1 < |z| < 2$.

(c) Solve the equation $z^4 + 1 = 0$. Find the residue of the function $f(z) = \dfrac{z^2+1}{z^4+1}$ at the single pole that lies in the first quadrant.

By using the contour of the quarter–circle form shown below, traced anticlockwise, show that

$$\int_0^\infty \frac{1+x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}} \qquad \text{and} \qquad \int_0^\infty \frac{1-x^2}{1+x^4}\,dx = 0.$$