UNIVERSITY OF ABERDEEN **SESSION 2003–04**

Degree Examination

MX3522 Complex Analysis

Thursday 27 May 2004 (3pm to 5pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Sketch the following subsets of the complex plane, giving a brief explanation of your results.

$$2 < |z-1| < 3, \qquad |z-3| = |z+i|\,.$$

(b) Find all solutions to the equation $\sin z = i$. You are told that the solution to the equation $\sinh y = 1$ is $y = \ln(1+\sqrt{2})$.

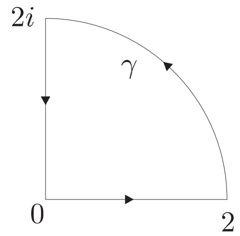
(c) Evaluate $\left|(1+i)^{i-1}\right|$, using the Principal Value for the power.

(d) Define what is meant by an open set in the complex plane. Prove that the union of two open subsets of the complex plane is an open set.

2. **(a)** Prove that the function $f(z) = e^{\overline{z}}$ is nowhere analytic.

(b) An entire function takes all its values on the imaginary axis. Prove that it is a constant function.

(c) Let γ be the contour shown in the diagram. It consists of the real axis from 0 to 2, followed by an arc of the circle $|z| = 2$ from 2 to $2i$, followed by the imaginary axis from $2i$ back to 0. Evaluate the following integrals, stating carefully any results that you use,



$$\int_\gamma \overline{z}\,dz, \qquad \int_\gamma e^z\,dz, \qquad \int_\gamma \frac{z^2+1}{(z-1-i)^2}\,dz, \qquad \int_\gamma \frac{z^4-1}{z-2-2i}\,dz.$$

3. **(a)** State Cauchy's Integral Formula giving $f'(p)$ at a point p inside a contour γ.

Let f be an entire function, p a point in the complex plane and $M(r)$ the maximum value of $|f(z)|$ on the circle $|z-p| = r > 0$. Prove that

$$|f'(p)| \leqslant \frac{M(r)}{r}.$$

State and prove Liouville's theorem.

(b) Let $f(z) = \dfrac{1}{z^2+z-6}$. Find the radius of convergence of the Taylor series of f expanded about the point $z = i$, stating clearly any result that you use.

By decomposing the function into partial fractions, find the coefficient of z^4 in the Taylor series of f expanded about $z = 0$.

4. **(a)** Define what is meant by the statements that a function f has

(i) a double pole at $z = 0$ with residue 3,

(ii) an isolated essential singularity at $z = 0$.

Give examples of functions that illustrate your definitions.

(b) Let $n > 1$ be an integer. Show that the function $f(z) = \dfrac{z^n}{z^n - 1}$ has n simple poles and find the sum of their residues.

(c) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{4+\cos\theta}.$$