UNIVERSITY OF ABERDEEN **SESSION 2006–07**

DEGREE EXAMINATION

MX3522 Complex Analysis

Wednesday 23 May 2007 (9 am to 11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Sketch the following subsets of the complex plane, giving a brief explanation of your results.

$$1 < |z + 2i| < 3, \qquad |z - 1| = |\overline{z} + 1|\,.$$

(b) Prove or provide a counterexample to each of the following statements (the Principal Value of the Logarithm is used).

$$\log(e^{2z}) = 2z \qquad \forall z \in \mathbb{C} \setminus \{0\}, \qquad e^{\log(z)} = z \qquad \forall z \in \mathbb{C} \setminus \{0\}$$

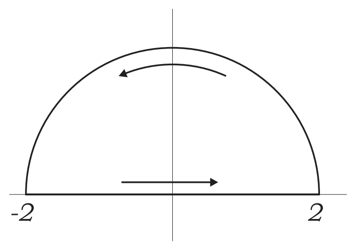
(c) Evaluate i^{1-i}, using the Principal Value for the power.

(d) Define what is meant by an *open set* in the complex plane. Prove that the following subset of $\mathbb{C}$ is open,

$$\mathcal{U} = \{x + iy \in \mathbb{C} : y > 0\}.$$

2. **(a)** Prove that the function $f(z) = \sin(\overline{z})$ is nowhere analytic.

(b) Let γ be the contour shown in the diagram. It consists of the real axis from -2 to 2, followed by the semicircle $|z| = 2$ from 2 to -2. Evaluate the following integrals, stating carefully any results that you use,



$$\int_\gamma \overline{z}\, dz, \qquad \int_\gamma e^{z^2}\, dz, \qquad \int_\gamma \frac{e^z}{(z+i)(z-i)^2}\, dz.$$

3. **(a)** A function f is analytic on and inside a contour γ (traced anti-clockwise). State Cauchy's Integral Formula giving the value of the derivative $f^{(k)}(p)$ at a point p inside γ.

Deduce that if f is an entire function then, given any point $p \in \mathbb{C}$,

$$\left|f^{(k)}(p)\right| \leqslant \frac{k!M(r)}{r^k},$$

where $M(r)$ is the maximum value of $|f(z)|$ on the circle $|z-p| = r > 0$.

Now suppose that $f(z)$ is an entire function and that f has the property that, for each $p \in \mathbb{C}$,

$$\left|\frac{f(z)}{(z-p)^2}\right| \to 0 \qquad \text{as} \quad |z-p| \to \infty.$$

By considering the second derivative of f, or otherwise, prove that $f(z) = az+b$ for some constants a and b.

(b) Let $f(z) = \dfrac{1}{(z+2)(z+i)}$. Find the radius of convergence of the Taylor series of f expanded about the point $z = 1+i$, stating clearly any result that you use.

Obtain the z^{-3} term in the Laurent series of $f(z)$, expanded about $z = \infty$ in negative powers of z, that is valid on the annulus $|z| > 2$.

4. **(a)** Each of the following functions has an isolated singularity at $z = 0$. In each case decide whether the singularity is removable, essential or is a pole. Find the residue at the singularity for each function.

$$\frac{e^z}{z^3}, \qquad \frac{1-\cos z}{3z^2}, \qquad e^{2/z}.$$

(b) Let γ be the positively oriented circle $|z-2i| = 2$. Use the Residue Theorem to evaluate the integral

$$\int_\gamma \frac{z^2}{1-z^4}\,dz.$$

(c) Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+2i\sin\theta} = \frac{2\pi}{\sqrt{13}}.$