UNIVERSITY OF ABERDEEN **SESSION 2003–04**

DEGREE EXAMINATION

MX3503 Linear Optimisation and Numerical Analysis

Monday 31 May 2004 (12noon to 2pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(i)** State the Karush-Kuhn-Tucker Theorem.

(ii) Consider the following optimisation problem: Find the minimum and maximum of $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 + z^2 = 1$, $x^2 + z^2 = 1$, $x \geqslant 0$, $z \geqslant 0$ and $x + y + z \leqslant 1$. Write the three Karush-Kuhn-Tucker equations for this problem.

(iii) Write all the systems of equations resulting from the Karush-Kuhn-Tucker theorem, corresponding to one inequality constraint being active, and the rest inactive. *Do not* attempt to solve the systems, but do simplify them as much as possible. (The number of equations must be the same as the number of variables.)

2. Consider the following optimisation problem: Find the minimum and maximum of the function $f(x, y) = x - y$, subject to the constraints $x^2 + y^2 \leqslant 1$, $2x^2 - 3y \leqslant 0$.

(i) Sketch the region defined by the constraints, and explain without performing any calculations, why f obtains a global minimum and a global maximum on the region defined by the constraints. Explain also why any local extremal point of f on the region is global. You may assume without proof that the region is convex.

(ii) Write the Karush-Kuhn-Tucker equations for this problem.

(iii) Use the Karush-Kuhn-Tucker method to find the global minimum and the global maximum of f subject to the given constraints.

3. **(a)** Define what it means for a subset $E \subseteq \mathbb{R}^n$ to be convex. Define also what it means for function $f\colon E \to \mathbb{R}$, defined on a convex subset $E \subseteq \mathbb{R}^n$, to be convex.

(b) Let $f\colon E \to \mathbb{R}$ be a convex function defined on a convex subset $E \subseteq \mathbb{R}^n$. Assume that f obtains a minimum on E and show that the set of all $\mathbf{x} \in E$, such that $f(\mathbf{x})$ is a local minimum of f on E, is a convex subset of $\mathbb{R}^n$, and that f is constant on it. (You may use the fact the any local minimum of a convex function is global.)

(c) Consider the functions

$$f(x, y) = 5x^2 + 2xy + y^2 - 6x + 2y + 7 \text{ and}$$

$$g(x, y) = x^2 + 4xy + 4y^2 + x - 2y.$$

Use the Hessian matrix to show that f is strictly convex and that g is convex over the entire xy-plane. Find the global extrema for f and g on the plane, if they exist.

4. **(a)** A pharmaceutical company is creating a tablet for a new drug. Each tablet is to contain three substances B, C and D, in addition to the active drug ingredient A, which is to be 14% of the weight of each tablet. Chemical and physical considerations mean that the weight of substance C should not exceed 25% of the combined weights of substance B and the active ingredient A, and that there should be at most 10 times as much of substance D as of substance B. The price per kilogram of substances B, C and D is £50, £15 and £2 respectively.

The problem is to decide how to formulate the tablet in order to minimise its cost. Express the problem as a linear programming problem. [You are *not* asked to solve the problem.]

(b) Consider the linear programming problem: Maximise $10 - x_1 - 4x_2 - 2x_3$ subject to $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ and

$$x_1 + 3x_2 + x_3 \leq 8,$$

$$x_1 - 2x_2 - x_3 \geq 6,$$

$$2x_1 - 3x_2 + x_3 \geq -2.$$

Show how to insert slack variables in such a way as to obtain the following tableau.

Basis	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{e}$	$\mathbf{b}$
$\mathbf{a}_4$	1	3	1	1	0	0	0	8
$\mathbf{a}_5$	-2	3	-1	0	1	0	0	2
$\mathbf{a}_6$	-1	2	1	0	0	1	0	-6
$\mathbf{e}$	1	4	2	0	0	0	1	10

Apply the simplex method to obtain a final tableau, and show that the optimal value is 4.