UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION

MX3502 Groups and Geometry

Wednesday 25 May 2005 (9 am—11 am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Let G be a group and let H be a subgroup of G. Define what is meant by a left coset of H in G, and by a right coset of H in G. Assuming that G is finite, prove that $|H|$ divides $|G|$ (Lagrange's theorem).

(b) Define what is meant by the symmetric group S_n. Define what is meant by the sign $\epsilon(\lambda)$ of a permutation $\lambda \in S_n$. Define the alternating subgroup A_n of S_n.

(c) List all elements of order $\leqslant 2$ in A_4 and prove that they form a subgroup H of A_4.

(d) Let $\sigma \in S_{12}$ be the permutation given in Cauchy notation by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 3 & 6 & 1 & 9 & 2 & 11 & 10 & 4 & 7 & 5 & 8 \end{pmatrix}.$$

Write σ in cycle notation. Determine $o(\sigma)$, the order of σ. Calculate σ^{15} and present it in cycle notation. Determine the orbits of σ. Determine the orbits of σ^{15}. Find $\epsilon(\sigma)$, the sign of σ.

(e) Find a generator of $(\mathbb{Z}/17)^*$, the multiplicative group of nonzero elements of the field $\mathbb{Z}/17$. List all elements of order 16 in $(\mathbb{Z}/17)^*$. List all elements of order 4 in $(\mathbb{Z}/17)^*$.

2. **(a)** Define what is meant by an action of a group G on a set X. Define what is meant by an orbit of the action. Define what is meant by the stabiliser subgroup $\mathrm{St}(y)$ for an element $y \in X$.

(b) Let H be the set of all invertible 2×2–matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are real numbers satisfying $a + 2b = 1$ and $c + 2d = 2$. Ordinary matrix multiplication makes this into a group. [You are not required to verify this.] There is a left action of H on $\mathbb{R}^2$ defined in the standard manner by multiplying matrices with column vectors. [You are not required to verify that the conditions for an action are satisfied.] Determine the orbits of this action. Choose an element in each orbit and determine its stabiliser subgroup.

(c) Define what is meant by the centre $Z(G)$ of a group G.

(d) Let G be a finite group of order p^n, where p is a prime number and $n > 0$. Prove that there exists $x \in G \setminus \{e\}$ which is central, that is, $x \in Z(G)$.

3. **(a)** Let G be a group with the neutral element e and let $x \in G$. Define what is meant by the order $o(x)$ of x.

Define what is meant by a group $\langle x \rangle$ generated by an element x.

Let x and y be elements of a group G. Prove that $o(xy) = o(yx)$.

Prove that $o(xyx^{-1}) = o(y)$ for all $x, y \in G$.

Prove that if $o(x^7) = 3$, then $o(x)$ equals 3 or 21.

Prove that if $o(x^3) = 3$, then $o(x) = 9$.

(b) Define what is meant by a subgroup of a group. In each of the following cases determine whether the set H is a subgroup of the group G. Justify your answers.

(i) $G = \mathbb{C}^*$ with multiplication of complex numbers,

$$H = \{z \in \mathbb{C}^* \mid -\pi/2 < \arg(z) < \pi/2\};$$

(ii) $G = GL_2(\mathbb{R})$ with multiplication of matrices,

$$H = \left\{ \begin{pmatrix} a & b \\ -2b & a \end{pmatrix} \in GL_2(\mathbb{R}) \mid a, b \in \mathbb{R} \right\};$$

(iii) $G = \mathbb{Q}$ with addition of rational numbers,

$$H = \left\{ x \in \mathbb{Q} \mid \exists n \in 1 + 10\mathbb{Z} \text{ such that } nx \in \mathbb{Z} \right\}.$$

(c) Regard $\mathbb{R}$ as a group using addition of real numbers. Describe the smallest subgroup H of $\mathbb{R}$ which contains the numbers $1/3$ and $1/7$. Is H a cyclic group? Justify your answer.

4. **(a)** Let F and G be groups. Define what is meant by a homomorphism from F to G. Define what is meant by the kernel and the image of such a homomorphism and what is meant by a normal subgroup of F.

Prove that the kernel of a homomorphism from F to G is a normal subgroup of F.

(b) Let G be a group, H a normal subgroup of G. Explain how the set of left cosets G/H can be made into a group. State the "first isomorphism theorem".

(c) Let N_1 and N_2 be normal subgroups of a group G with $N_1 \cap N_2 = \{e_G\}$. Suppose $x \in N_1$, $y \in N_2$. Show that $xy = yx$.

(d) Let G be a cyclic group of order 7. Determine the number of homomorphisms from G to the symmetric group S_7.